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Unsold Versus Unbought Commitment: Minimum Total Commitment Contracts with Nonzero Setup Costs

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We study a minimum total commitment (MTC) contract embedded in a finite-horizon periodic-review inventory system. Under this contract, the buyer commits to purchase a minimum quantity of a single product from the supplier over the entire planning horizon. We consider nonstationary demand and per-unit cost, discount factor, and nonzero setup cost. Because the formulations used in existing literature are unable to handle our setting, we develop a new formulation based on a state transformation technique using unsold commitment instead of unbought commitment as state variable. We first revisit the zero setup cost case and show that the optimal ordering policy is an unsold-commitment-dependent base-stock policy. We also provide a simpler proof of the optimality of the dual base-stock policy. We then study the nonzero setup cost case and prove a new result, that the optimal solution is an unsold-commitment-dependent (s, S) policy. We further propose two heuristic policies, which numerical tests show to perform very well. We also discuss two extensions to show the generality of our method's effectiveness. Finally, we use our results to examine the effect of different contract terms such as duration, lead time, and commitment on buyer's cost. We also compare total supply chain profits under periodic commitment, MTC, and no commitment.

Key words: supply contracts; inventory management; quantity commitment; dynamic programming; (s, S) policy
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1. Introduction

The procurement function has been identified as a key driver for competitiveness and the design of supply contracts has become an important area of study. At its core, a supply contract is a binding mechanism to properly allocate risks between the upstream supplier and the downstream buyer. The question that supply contracts address is who should bear the risk of producing ahead of time and keeping inventories of raw materials, components, and finished goods. On one hand, the supplier would prefer stability whereby the buyer is required to place an order ahead of time, pay, and take ownership of the goods. On the other hand, buyers like the flexibility to place orders and then change these orders as they observe more information about demand, hence delaying ownership to the last minute. Supply contracts are put in place precisely to strike a balance between upstream stability and downstream flexibility. For example, by initiating a so-called minimum total commitment (MTC) contract with its supplier, Stanley Black & Decker Co. Ltd is able to cut lead times from 4 weeks to 1 week (Zou 2012). This provides upstream stability because the supplier is more assured of demand and hence can allocate capacity and associated resources with confidence. In return, the supplier is willing to provide downstream flexibility in the form of shorter lead times. Moreover, MTC contracts have also been observed in other industries such as electronics (Campbell 2012), aviation (Maxon 2013), pharmaceutical (Business Wire 2013), and mining (Tally Metal Sales 2014).

In this study, we examine this class of supply contracts called MTC contracts. Specifically, this is a contract made in advance under which the buyer must order at least a certain number of units from the supplier over the contract period (e.g., 1 year). In exchange, the supplier agrees to sell the product at a predetermined (usually discounted) price or with a predetermined (usually shorter) lead time. The MTC

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contract provides flexibility to the buyer to decide when to consume each unit of the total commitment, as opposed to a forward contract where order quantities for all replenishment periods are fixed in advance. It also reduces uncertainty for the supplier who now faces less downside risk for total demand during the contract period, providing more incentive to invest in capacity. In this study, we consider a periodic review inventory model and study the optimal ordering policy when there is a MTC to be fulfilled by the end of the planning horizon (i.e., the contract period). The results of our analysis can then be used to evaluate and compare various MTC contracts with different contract terms (e.g., duration, MTC, component cost, lead time). In other words, our results will enable us to make trade-offs among the various contract terms toward better contract design.

Production-inventory models incorporating supply contracts with various forms of quantity commitments have been studied in the literature. Bassok and Anupindi (1997) consider the MTC contract in a single supplier, single buyer supply chain. In this supply chain, the buyer must order up to a minimum total order quantity by the end of a finite contract horizon. The optimal policy is characterized by a dual base-stock policy with two critical order-up-to levels. The two critical levels are obtained from the corresponding finite horizon and a single-period standard newsvendor problem with no commitment, but with discounted price. However, their model only considers stationary demand, stationary per-unit cost, no discount factor, and zero setup cost. Chen and Krass (2001) extend Bassok and Anupindi's (1997) work to the case where demand is nonstationary and per-unit cost is still stationary but different beyond the MTC. They show that the optimal ordering policy remains the dual base-stock type. Anupindi and Bassok (1998) also extend Bassok and Anupindi (1997) to the multi-product setting. In this contract, the MTC is specified in terms of total amount of dollars to be purchased by the end of the contract period. Feng and Sethi (2010) examine the interaction between supply price uncertainty and demand uncertainty using different procurement options such as long-term price-only contracts, capacity reservation contracts, and spot market purchases. For the second type, they also study reserved overall capacity, which is an upper bound on total order quantity. Peng et al. (2012) also consider overall capacity reservation in the three-layer contract negotiation process they modeled for Intel. In the execution layer, they investigate a dynamic dual-source capacity expansion problem with total capacity constraint over the entire planning horizon. Due to problem complexity, they utilize a heuristic to solve this layer which is then used to analyze the other two layers. Feng et al. (2013) study a three-tier supply chain motivated by a case study in the Oriented Strand Board (OSB) industry, where a manufacturer contracts with both downstream customers and upstream suppliers. While they consider periodic commitment for downstream contracts, they use MTC contracts upstream. The focus of their paper is not to characterize the optimal policy but to arrive at numerical solutions (e.g., using Sampling Approximation Approach).

There have been many justifications for purchase commitments in supply contracts in the literature. The model of Anupindi and Akella (1993) is among the first of its kind: the buyer agrees to accept delivery of a fixed quantity of goods in each period of the contract duration, whereas the buyer faces uncertain periodic demand. While the supplier delivers the pre-contracted quantity immediately in each period, additional units might be shipped with positive lead-times. Henig et al. (1997) consider a supply contract over a finite horizon whereby a minimum order quantity is imposed on the buyer in each period at a predetermined (and prepaid) cost. Replenishments in excess of the minimum quantity incur additional costs. They find that the optimal ordering policy has two critical levels and there is a range of inventory levels for which the quantity ordered equals the contract volume. Moinzadeh and Nahmias (2000) consider a multi-period contractual agreement between buyer and seller in which Q units are delivered to the buyer at regular time intervals. Facing uncertain market demand, the buyer has the option of adjusting the delivery quantity upwards just prior to a delivery, but must pay a premium to do so. They also show that the fixed delivery contract results in lower variance of orders to the seller. Hence, such a contract serves as a risk-sharing mechanism. Note, however, that the above works assume periodic commitment, that is, a purchase commitment in each period (or interval). That said, for further justification on the use of purchase commitments, the reader is encouraged to see Anupindi and Bassok (1999) and Tsay et al. (1999).

There are a number of other works on supply contracts involving some form of flexible commitment or minimum order quantity. Tsay and Lovejoy (1999) consider a quantity flexibility contract which stipulates a maximum percentage revision allowed for the order quantity in each period over a finite planning horizon. To solve the problem, the authors use a heuristic approach to transform the original stochastic problem into a deterministic one. Urban (2000) models supply contracts with fixed and stationary periodical commitment and limited flexibility to change the order quantity at a cost to the buyer. The multiple newsvendor periods are connected by their stationary (hence common) commitment quantity. He then provides a solution methodology to solve the

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problem. Sethi et al. (2004) study quantity flexibility contracts with demand forecast updating and availability of a spot market. They characterize the optimal ordering policy and discuss the impact of forecast quality and degree of contract flexibility on the optimal decisions. Martinez-de-Albeniz and Simchi-Levi (2005) use a portfolio approach to show that a quantity flexibility contract is equivalent to a combination of a forward buy contract and an option contract. Ben-Tal et al. (2005) analyze the optimal replenishment decisions under flexible commitments contracts using the robust optimization approach where uncertain demand is only known to reside in some uncertainty set. Altintas et al. (2008) consider a supply contract that offers the buyer an all-units quantity discount. They study the optimal ordering policy for the buyer and provide insights to the supplier on how to set effective discount parameters. Bassok and Anupindi (2008) address rolling horizon flexibility contracts whereby at the beginning of the horizon, the buyer commits requirements for components for each period into the future. Meanwhile, the supplier provides limited flexibility to the buyer to adjust the current order and future commitments in a rolling horizon manner. They propose two heuristics to obtain good procurement decisions for the buyer. Scheller-Wolf and Tayur (2009) study not only minimum order quantities but also order capacities in each period, together referred to as order bands. They use infinitesimal perturbation analysis to optimize the class of base-stock policies and show that using order bands is an effective risk-sharing mechanism. Lian and Deshmukh (2009) examine a supply contract under which the buyer receives discounts for committing to purchase in advance such that the further in advance the commitment is made, the larger the discount. They obtain the optimal ordering policy for this contract over a rolling horizon with demand forecast updating. Similar to our problem, the above papers consider some form of flexible commitment or minimum order quantity in a multi-period setting. However, they only consider periodic commitments or minimum quantities per period while our problem imposes a minimum amount on the total replenishment over the entire contract duration.

Another stream of related research looks at one-period or two-period models involving flexible order quantities. Eppen and Iyer (1997) study the replenishment policy under a backup contract, which allows the retailer to return a portion of its purchase to the supplier. They derive the optimal solution and find that backup contracts can have a substantial impact on expected profits and may result in a higher committed quantity. Tsay (1999) models the incentives of both the supplier and the buyer in a quantity flexibility contract. He shows that properly designing the contract can coordinate the channel to achieve the system-wide optimal outcome. Taylor (2001) also studies a two-period problem where the retailer can either order more or return part of excess stock to the supplier at the beginning of the second period. Barnes-Schuster et al. (2002) model quantity flexibility through options for a two-period model with correlated demand. They show how options can provide flexibility for quick response to market changes and can also coordinate the channel. More recently, Durango-Cohen and Yano (2011) study a forecast-commitment contract where the buyer provides a forecast for a future order and a guarantee to purchase a portion of it, while the supplier commits to satisfy some or all of the forecast. The supplier pays penalties for shortfalls of the commitment quantity from the forecast and for shortfalls of the delivered quantity from the buyer's order quantity. However, as mentioned, these works look at models of only one period or two periods, and also focus on issues fundamentally different from ours (e.g., channel coordination).

As hinted above, our work in this study is most closely related to Bassok and Anupindi (1997) and Chen and Krass (2001). While these two earlier works do not consider nonstationary per-unit cost, discount factor, and nonzero setup cost, we will do so in this study. Table 1 summarizes the similarities and differences among the three papers.

That said, we find that the formulations adopted in the earlier papers cannot be extended to handle our setting, particularly in characterizing the optimal policy for the nonzero setup cost case. To fix this problem, we introduce a new formulation that hinges on an intuitive yet powerful state transformation technique. While a similar transformation technique was used in Yuan et al. (2013), this study is the first to use it in the MTC setting[1] and to managerially interpret it as a distinction between "unbought commitment"

Table 1 Most Closely Related Literature

	Demand	Per-unit cost	Discount factor	Setup cost
Bassok and Anupindi (1997)	Stationary	Stationary	No	Zero
Chen and Krass (2001)	Nonstationary	DBC*	Yes†	Zero
This paper	Nonstationary	Nonstationary	Yes	Nonzero

*DBC means different beyond the commitment.
†Only if per-unit cost within commitment is no greater than that beyond commitment.

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and "unsold commitment." Unbought commitment refers to the number of units in the MTC that are still with the supplier, while unsold commitment refers to the number of units in the total commitment that are not yet sold to the customers, regardless of whether the units are with the supplier or the buyer. The earlier papers utilize unbought commitment as one of two state variables, the other being on-hand inventory. This creates a problem for the nonzero setup cost case due to the difficulty in handling K-convexity in multiple dimensions.

Our new formulation replaces unbought commitment with unsold commitment as the state variable. We find that while unbought commitment is dependent on inventory decisions, unsold commitment only depends on realized demands. Hence, at any period, the unsold commitment is merely the MTC less the cumulative demand up to that point. This property allows us to show that the value function evolves exogenously regardless of the inventory decisions, and it becomes separable in the two state variables under certain conditions. We are hence able to characterize the optimal ordering policy for nonzero setup cost, which turns out to be an unsold-commitment-dependent (s, S) policy.

Our work makes the following contributions to the literature. We first introduce the notion of "unsold commitment" in the MTC setting. We demonstrate how our new formulation works in the case of zero setup cost but unlike existing literature, we consider nonstationary per-unit cost and a discount factor. To this end, we manage to show that the optimal ordering policy is an unsold-commitment-dependent base-stock policy. We also provide a simpler proof of the optimality of the dual base-stock policy. Then, we use the new formulation to characterize the optimal ordering policy for nonzero setup cost, which previous formulations cannot handle. We prove for the first time the optimality of an unsold-commitment-dependent (s, S) policy. We also propose two heuristic policies for the nonzero setup cost case and numerically test their performance vis-a-vis the optimal policy. Next, we extend our approach and analysis to two cases; namely, per-unit cost different beyond commitment and nonzero lead time. We find that the optimal policy structures carry over to these scenarios. Finally, we examine the effect of changing contract terms such as contract duration, lead time, and total commitment on the buyer's optimal cost. We also compare total supply chain profits under periodic commitment, MTC and no commitment.

The remainder of this study is organized as follows. Section 2 introduces our model formulation and makes preparations for model analysis with some preliminary results. We then characterize, in section 3, the optimal ordering policy for the zero setup cost case. Section 4 presents our analysis of the optimal ordering policy and a heuristic policy for the case of nonzero setup cost. In section 5, we present the two extensions. Section 6 examines buyer's cost as contract terms change, while section 7 looks into total supply chain profits. Finally, Section 8 concludes while omitted technical proofs can be found in the Online Appendix.

2. Model Formulation and Preliminary Results

In this section, we first present the basic model. Then, we introduce the notion of "unsold commitment" which we shall use to show an equivalent transformation of the model formulation. Finally, we present two properties that are sufficient conditions for the value function to be separable in the state variables. These results will form part of the foundation of our structural model analysis in subsequent sections.

2.1. Model Description

Consider a single-item inventory system in discrete time embedded in a supply contract. The contract duration, which is also the planning horizon, is assumed to be finite with T periods. At the beginning of the planning horizon, that is, at time $t = 1$, the buyer commits to purchasing a minimum quantity Q_1^r over the entire horizon. The firm only needs to fulfill the commitment by the end of the planning horizon, that is, at time $t = T + 1$, and may place an order of any quantity in every period. This means that the buyer does not have to purchase the entire committed quantity in one period. Therefore, the contract is time-flexible.

We assume that the buyer is risk neutral, thus he aims to find a replenishment policy that minimizes the expected value of the total discounted costs over the planning horizon. The relevant costs include fixed order setup cost and per-unit ordering cost, as well as inventory (holding and backordering) costs. In each period t, the sequence of events is as follows.

1. At the beginning of the period, the buyer observes the initial on-hand inventory level x_t. If $x_t < 0$, then it means that there were units of unmet demand from the previous period that were backordered to this period. The buyer also observes the remaining "unbought commitment" Q_t^r, that is, the remaining quantity that the buyer has yet to fulfill at the beginning of period t. If $Q_t^r \leq 0$, then it means that the MTC specified in the contract has already been fulfilled.
2. The buyer may place an order to bring the inventory level up to $y_t \geq x_t$. If $y_t > x_t$, then

a positive order quantity $y_t - x_t$ is placed, in which case a fixed setup cost $K \geq 0$ is incurred, as well as a per-unit cost of c_t for each unit ordered. We assume that lead time for order delivery is zero, without loss of generality (see further discussion for nonzero lead time in section 5, Extension II).

3. Customer demand D_t is realized. We further assume that $D_t, t = 1, 2, \ldots, T$, are independent, nonnegative random variables and bounded above by $\bar{D}$. Demand in the period is fulfilled as much as possible by on-hand inventory, and any unmet demand will be fully backordered with a backordering cost. Otherwise, excess inventory is carried over with a holding cost to the next period.
4. Inventory costs are assessed and incurred. Denote by $L_t(y)$ the expected one-period convex inventory (holding and backordering) cost if the buyer orders up to y in period t. Moreover, let $L_t(y) = h_t \cdot \mathbb{E}[\max\{y - D_t, 0\}] + b_t \cdot \mathbb{E}[\max\{D_t - y, 0\}]$, where h_t and b_t are the holding cost rate and backordering cost rate, respectively, in period t.
5. Time ages to the next period. If next period is the end of the planning horizon, that is, $t = T + 1$, then the buyer purchases all the remaining unbought commitment as agreed upon in the contract terms.

As in existing literature, we formulate our problem as a dynamic program. Assume that all costs in future periods are discounted at a rate of $\alpha \leq 1$. The total discounted expected minimum cost from time t to time $T + 1$ is denoted by $\bar{v}_t(x_t, Q_t^r)$, with two state variables; namely, inventory level x_t at the beginning of period t before ordering, and remaining unbought commitment Q_t^r. The buyer's goal is to control inventory decisions, y_t, to minimize the total expected cost in time t. Hence, we can write the dynamic programming recursion for $\bar{v}_t(x_t, Q_t^r)$ as

$$\begin{aligned}\bar{v}_t(x_t, Q_t^r) = &\min_{y_t \geq x_t}\{K\delta(y_t - x_t) + c_t(y_t - x_t) + L_t(y_t)\\ &+ \alpha\mathbb{E}[\bar{v}_{t+1}(y_t - D_t, Q_t^r - (y_t - x_t))]\}\end{aligned} \tag{1}$$

for t = 1, 2, 3, …, T, where

$$\delta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Moreover, the terminal function is

$$\begin{aligned}\bar{v}_{T+1}(x_{T+1}, Q_{T+1}^r) = K\delta(y_{T+1} - x_{T+1}) + c_{T+1}(y_{T+1}\\ - x_{T+1}),\end{aligned} \tag{2}$$

$$y_{T+1} = \max\{Q_{T+1}^r + x_{T+1}, x_{T+1}, 0\}. \tag{3}$$

The terminal condition Equation (3) ensures that all outstanding unbought commitment Q_{T+1}^r will be fulfilled by $T + 1$, and that all backorders will also be cleared. Clearly, when Q_{T+1}^r is negative while x_t is nonnegative, there will be no ordering in $T + 1$.

From formulation Equation (1), we can also infer the transition of the state variables as follows.

$$x_{t+1} = y_t - D_t \tag{4}$$

$$Q_{t+1}^r = Q_t^r - (y_t - x_t), \tag{5}$$

where the decision variable y_t and states variables Q_t^r and x_t appear simultaneously in the second state transition equation. This causes difficulty in analyzing the dynamic program. To remedy this problem, we introduce a state transformation technique that affords a reformulation of the model in Equation (1) – (3) into something more analytically tractable.

2.2. The State Transformation Technique

We now introduce a new state variable as follows. Let

$$Q_t = Q_t^r + x_t, \quad t = 1, 2, \ldots, T. \tag{6}$$

It follows that the state transition equation (5) becomes

$$\begin{aligned}Q_{t+1} = Q_{t+1}^r + x_{t+1} &= (Q_t^r - (y_t - x_t)) + (y_t - D_t)\\ &= Q_t^r + x_t - D_t = Q_t - D_t.\end{aligned} \tag{7}$$

From here onwards, we shall refer to this new state variable as remaining "unsold commitment," which is the number of units in the MTC that are not yet sold to the customers, regardless of whether the units are with the supplier or with the buyer. This quantity contrasts with remaining unbought commitment according to the following relationship.

$$\begin{aligned}\text{Unsold Commitment} = &\text{ Unbought Commitment}\\ &+ \text{On-Hand Inventory}.\end{aligned}$$

More importantly, it is easy to see that at any period t, there is a one-to-one correspondence between (x_t, Q_t) and (x_t, Q_t^r). This observation allows us to replace Q_t^r with Q_t in the original formulation in Equation (1) – (3). We can, therefore, rewrite the dynamic programming formulation as follows.

$$\begin{aligned}\tilde{V}_t(x_t, Q_t) = &\min_{y_t \geqslant x_t}\{K\delta(y_t - x_t) + c_t(y_t - x_t) + L_t(y_t)\\ &+ \alpha\mathbb{E}[\tilde{V}_{t+1}(y_t - D_t, Q_t - D_t)]\}\end{aligned} \tag{8}$$

with terminal conditions

$$\tilde{V}_{T+1}(x_{T+1}, Q_{T+1}) = K\delta(y_{T+1} - x_{T+1}) + c(y_{T+1} - x_{T+1}), \tag{9}$$

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$$y_{T+1} = \max\{Q_{T+1}, x_{T+1}, 0\}. \quad (10)$$

Observe that the transition of the new state variable Q_t from period to period now depends only on demand D_t and is independent of inventory decision y_t. Moreover, as mentioned earlier, (x_t, Q_t) and (x_t, Q_t^r) form a one-to-one mapping. This implies that once we know the optimal strategy for Q_t, we can easily find the optimal strategy for Q_t^r. Hence, we can now focus on solving model Equation (8) – (10). To further simplify our analysis, we define

$$V_t(x_t, Q_t) = c_t x_t + \tilde{V}_t(x_t, Q_t),$$

$$H_t(y_t) = (c_t - \alpha c_{t+1})y_t + \alpha c_{t+1}\mathbb{E}D_t + L_t(y_t).$$

for $t = 1, 2, \ldots, T$. It follows that formulation Equation (8) – (10) can be re-expressed as

$$V_t(x_t, Q_t) = \min_{y_t \geqslant x_t}\{K\delta(y_t - x_t) + G_t(y_t, Q_t)\}, \quad (11)$$

$$G_t(y_t, Q_t) = H_t(y_t) + \alpha\mathbb{E}[V_{t+1}(y_t - D_t, Q_t - D_t)]. \quad (12)$$

with boundary conditions

$$V_{T+1}(x_{T+1}, Q_{T+1}) = K\delta(y_{T+1} - x_{T+1}) + c_{T+1}y_{T+1}, \quad (13)$$

$$y_{T+1} = \max\{Q_{T+1}, x_{T+1}, 0\}. \quad (14)$$

That said, in our model analysis in sections 3 and 4, we will show that the transformed value functions in Equation (11) – (14) are separable under certain conditions, which will make the structure of the optimal policy easier to analyze. In the next subsection, we first introduce two properties that are sufficient conditions for separable functions and then give some preliminary results about separable functions.

2.3. Sufficient Conditions for Separable Functions

For completeness, we provide the following definition of separable functions, as well as two properties to illustrate sufficient conditions for separable functions.

Definition 1. Suppose $\Omega \subseteq \mathbb{R}^2$. A function $f : \Omega \to \mathbb{R}$ is called *separable in* Ω if there exist functions $h : \mathbb{R} \to \mathbb{R}$ and $l : \mathbb{R} \to \mathbb{R}$, such that $f(x, Q) = h(x) + l(Q)$.

Definition 2. Define two properties:
Property I. *For any $\gamma > 0$ and for any x_2, if $Q_1 \leq Q_2$ and $Q_1 \leq x_1$, then*

$$f(x_1, Q_1) - f(x_1, Q_1 - \gamma) \leq f(x_2, Q_2) - f(x_2, Q_2 - \gamma). \quad (15)$$

Property II. *For any $\gamma > 0$ and for any Q_2, if $x_1 \leq x_2$ and $Q_1 \geq x_1$, then*

$$f(x_1, Q_1) - f(x_1 - \gamma, Q_1) \leq f(x_2, Q_2) - f(x_2 - \gamma, Q_2). \quad (16)$$

With Property I and Property II, we can now prove the separable property of functions when $x \geq Q$ and $x \leq Q$, respectively, in the following result.

Lemma 1. *If f satisfies Property I, then f(x, Q) is separable when $Q \leq x$; if f satisfies Property II, then f(x, Q) is separable when $Q \geq x$.*

Next, we present the following results about separable functions, which will be useful for our analysis in the next sections.

Lemma 2. *Suppose $f_1(y, Q) : \mathbb{R}^2 \to \mathbb{R}$ and $f_2(y, Q) : \mathbb{R}^2 \to \mathbb{R}$ are continuous separable functions and both are convex in y for any Q. For a given $x \in \mathbb{R}$, we define*

$$S_1^*(Q) = \arg\min_{y\in\mathbb{R}}\{f_1(y, Q)\};$$
$$S_2^*(Q) = \arg\min_{y\in\mathbb{R}}\{f_2(y, Q)\};$$
$$y_1^*(x, Q) = \arg\min_{y\geqslant x,\ y\geqslant Q}\{f_1(y, Q)\};$$
$$y_2^*(x, Q) = \arg\min_{y\geqslant x,\ y\leqslant Q}\{f_2(y, Q)\}.$$

Then, the following properties hold.

(a) $S_1^*(Q) = S_1^*$ and $S_2^*(Q) = S_2^*$, that is, they are constant and independent of Q.
(b) The optimal values of $y_1^*(x, Q)$ and $y_2^*(x, Q)$ are given as

$$y_1^*(x, Q) = \max(S_1^*, Q, x)$$
$$y_2^*(x, Q) = \begin{cases} Q, & \text{if } x \leq Q \leq S_2^*\ ; \\ S_2^*, & \text{if } x \leq S_2^* \leq Q\ ; \\ x, & \text{if } S_2^* \leq x \leq Q\ ; \\ \text{undefined}, & \text{otherwise} \end{cases}$$

(c) Suppose $f_1(Q, Q) = f_2(Q, Q)$ and $S_1^* \leq S_2^*$. Then,

$$g(y, Q) = \begin{cases} f_1(y, Q), & \text{if } y \geq Q\ ; \\ f_2(y, Q), & \text{if } y < Q \end{cases}$$

is continuous, and separable when $y \geq Q$ and separable when $y < Q$. Moreover,
if $y_g^*(x, Q) = \arg\min_{y\geqslant x}\{g(y, Q)\}$, then

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$$y_g^*(x,Q) = \begin{cases} S_1^*, & \textit{if } Q \le S_1^* \textit{ and } x \le S_1^*\ ; \\ Q, & \textit{if } S_1^* \le Q \le S_2^* \textit{ and } x \le Q\ ; \\ S_2^*, & \textit{if } Q \ge S_2^* \textit{ and } x \le S_2^*\ ; \\ x, & \textit{otherwise.} \end{cases}$$

3. Revisiting the Case of Zero Setup Cost

In this section, we re-examine the case with zero fixed setup cost, that is, $K = 0$. While the optimality of the dual base-stock policy for this case has been well established in the literature (Bassok and Anupindi 1997; Chen and Krass 2001), our proposed formulation allows us to achieve two new results. First, we prove that the optimal policy is an unsold-commitment-dependent base-stock policy. Second, we provide a simpler proof of the optimality of the dual base-stock policy. That said, the basic model Equation (11) – (14) can be written as follows.

$$V_t(x_t,Q_t) = \min_{y_t \geqslant x_t}\{G_t(y_t,Q_t)\}, \quad (17)$$

$$G_t(y_t,Q_t) = H_t(y_t) + \alpha\mathbb{E}[V_{t+1}(y_t - D_t, Q_t - D_t)], \quad (18)$$

and the boundary condition is

$$V_{T+1}(x_{T+1},Q_{T+1}) = c_{T+1}\max\{Q_{T+1},x_{T+1},0\}. \quad (19)$$

We first present the following theorem which states that the optimal ordering policy is an unsold-commitment-dependent base-stock policy. This theorem is made possible largely because our formulation has the nice property that unsold commitment (unlike unbought commitment) evolves exogenously regardless of the inventory decision.

Theorem 3. *Suppose $K = 0$. Then, for $t = 1,2,\ldots, T+1$, both V_t and G_t are jointly convex. Moreover, the optimal ordering decision follows a base-stock policy and the optimal base-stock level depends on Q_t.*

However, Theorem 3 does not specify how the optimal base-stock level depends on Q_t. This will be addressed in the next results.

Lemma 4. *When $K = 0$, for any $t = 1,2,\ldots,T$, G_t, V_t and V_{T+1} satisfy Property I and Property II.*

By Lemmas 1 and 4, G_t is separable when $y \geq Q$ and also when $y < Q$. We denote

$$G_t(y_t,Q_t) = \begin{cases} G_t^1(y_t,Q_t), & \text{if } y_t \ge Q_t\ ; \\ G_t^2(y_t,Q_t), & \text{if } y_t < Q_t \end{cases}$$

Then, $G_t^1(y_t,Q_t)$ and $G_t^2(y_t,Q_t)$ are both separable. Let $S_t(Q_t)$ and $S_t^M(Q_t)$ be the global minimizers of $G_t^1(y_t,Q_t)$ and $G_t^2(y_t,Q_t)$, respectively. That is, $S_t(Q_t) = \arg\min_{y_t}\{G_t^1(y_t,Q_t)\}$ and $S_t^M(Q_t) = \arg\min_{y_t}\{G_t^2(y_t,Q_t)\}$. By Lemma 2(a), $S_t(Q_t) = S_t$ and $S_t^M(Q_t) = S_t^M$ are both constant in Q_t. Next, we will characterize S_t and S_t^M.

Definition 3. In period t, for $t = 1,2,\ldots,T$, we define two inventory models, denoted as (*P1*) and (*P2*), which will help us characterize S_t and S_t^M:

(*P1*) $v_t(x_t) = \min_{y_t \geqslant x_t}\{H_t(y_t) + \alpha\mathbb{E}v_{t+1}(y_t - D_t)\}$, with boundary condition $v_{T+1}(x_{T+1}) = c_{T+1}\max\{x_{T+1}, 0\}$.
(*P2*) $u_t(x_t) = \min_{y_t \geqslant x_t}\{H_t(y_t) + \alpha\mathbb{E}u_{t+1}(y_t - D_t)\}$, with boundary condition $u_{T+1}(x_{T+1}) = 0$.

In (*P1*), $v_t(x_t)$ refers to the value function of the classic T-period newsvendor model. For all t, if we set $Q_t \le x_t$ in our model Equation (17) – (19), that is, there is no remaining unbought commitment, it will be proved that the optimal solution of our problem is identical to the unconstrained optimal solution of this standard inventory model.

In (*P2*), $u_t(x_t)$ represents the value function in the case that we commit to ordering a fixed amount of quantity Q so large that the commitment cannot be fulfilled until period $T + 1$. We let the last-period value with respect to inventory be $u_{T+1}(x_{T+1}) = 0$, which has no relations with x_{T+1}. If the commitment quantity in the last period is larger than zero, it is clear that the unbought quantity in each period must be purchased either in that or later periods. Therefore, it can be considered a sunk cost and the only relevant costs at this point in time are the inventory holding and backordering costs.

We now present the characterization of S_t and S_t^M

Lemma 5. *For any period t, $t = 1,2,\ldots,T+1$,*

(a) *When $x_t \ge Q_t$, the optimal solution of $\min_{y_t}\{G_t^1(y_t,Q_t)\}$ is identical to the optimal solution of (P1): $S_t = \sup\{S_t^* | S_t^* \in \arg\min_{y_t}\{H_t(y_t) + \alpha\mathbb{E}v_{t+1}(y_t - D_t)\}\}$;*
(b) *When $x_t < Q_t$, the optimal solution of $\min_{y_t}\{G_t^2(y_t,Q_t)\}$ is identical to the optimal solution of (P2): $S_t^M = \sup\{S_t^{M*} | S_t^{M*} \in \arg\min_{y_t}\{H_t(y_t) + \alpha\mathbb{E}u_{t+1}(y_t - D_t)\}\}$.*

Remark 1. Our proof for Lemma 5, part (b) is simpler and more intuitive than those in the existing literature because separability in our value function allows us to find S_t^M by fixing Q_t to values convenient for analysis. This separable property does not hold for value functions in previous works.

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Before proceeding, we compare the two thresholds in the following lemma.

LEMMA 6. *For any* $t = 1, 2, \ldots, T, S_t \leq S_t^M$. *Moreover, for any* $t = 1, 2, \ldots, T + 1, v_t'(x) - u_t'(x) \geq 0$ *for any given x.*

We are now ready to present our main result on the optimal ordering policy for the case of zero setup cost.

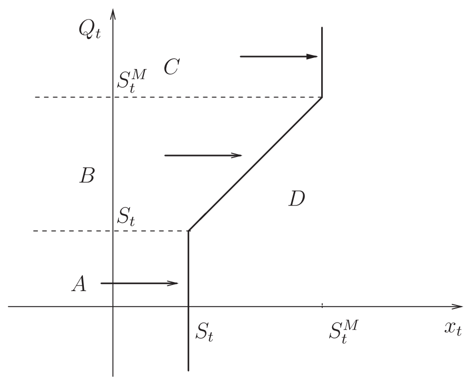
THEOREM 7. *In period* $t, t = 1,2,\ldots,T$, *the optimal order-up-to level* $y_t^*(x, Q)$ *can be characterized by the following form:*

$$y_t^*(x_t, Q_t) = \begin{cases} S_t, & \text{if } Q_t \leq S_t \text{ and } x_t \leq S_t\ ; & (A) \\ Q_t, & \text{if } S_t \leq Q_t \leq S_t^M \text{ and } x_t \leq Q_t\ ; & (B) \\ S_t^M, & \text{if } Q_t \geq S_t^M \text{ and } x_t \leq S_t^M\ ; & (C) \\ x_t, & \text{otherwise.} & (D) \end{cases}$$

PROOF. We consider the optimal policy in any period t, $t = 1, 2, \ldots, T$. From Theorem 3, Lemma 4, and Lemma 6, we have proven that $G_t(y_t, Q_t), V_t(x_t, Q_t)$ are all separable and convex functions, and that $S_t \leq S_t^M$. Therefore, our value function satisfies the assumptions in Lemma 2. Hence, the results easily follow. □

Theorem 7 provides an alternative presentation and a simpler proof of the optimality of the dual base-stock policy first established in Bassok and Anupindi (1997). It describes exactly how the optimal base-stock level mentioned in Theorem 3 depends on unsold commitment, that is, the optimal base-stock level is $\max\{\min(Q_t, S_t^M), S_t\}$. To end this section, we illustrate in Figure 1 the dual base-stock policy as a function of state variables x_t and Q_t. Observe that the arrows all move horizontally, because there is no change in unsold commitment Q_t even after a positive order is placed.

Figure 1 Optimal Policy for *K* = 0



4. The Case of Nonzero Setup Cost

In this section, we consider the case of nonzero setup cost, that is, $K > 0$. To the best of our knowledge, ours is the first paper to analytically examine the optimal policy for this case. With our proposed formulation, the problem becomes more tractable. However, while many inventory problems with nonzero setup cost can be handled by *K*-convexity and policies for these problems are well known, such methodology can only be applied to one-dimensional problems. Specifically, the preservation properties of *K*-convexity no longer hold in multidimensional problems (see Gallego and Sethi 2005). Since there are two state variables in model Equation (11) – (13), the standard *K*-convexity cannot be preserved. In the first subsection, we explore a new property of *K*-convexity, which allows us to prove the optimality of an unsold-commitment-dependent (*s*, *S*) policy. In the second subsection, we propose an easy-to-implement heuristic policy and numerically test its performance vis-a-vis the optimal policy.

4.1. Structural Analysis of Optimal Policy

We first present the following properties of *K*-convex functions.

LEMMA 8. *Suppose f(y,Q) is K-convex in y for any given Q.*

(a) *If W is a random variable, then* $\mathbb{E}[f(y - W, Q - W)]$ *is K-convex in y, provided* $\mathbb{E}[|f(y - W, Q - W)|] < \infty$ *for all (y,Q).*

(b) *If* $f(y,Q) \to \infty$ *as* $|y| \to \infty$, *then there exist s(Q) and S(Q) with s(Q) < S(Q) such that*
   (i) $f(S(Q),Q) < f(y,Q)$, *for all y;*
   (ii) $f(S(Q),Q) + K = f(s(Q),Q) < f(y,Q)$, *for all* $y < s(Q)$;
   (iii) $f(y,Q)$ *is decreasing in y on* $y \in (-\infty, s(Q))$;
   (iv) $f(y,Q) \leq f(z,Q) + K$ *for all y, z with* $s(Q) \leq y \leq z$.

PROOF. For part (a), $f(y,Q)$ is *K*-convex in *y*, then $f(y - w, Q - w)$ is *K*-convex in *y* for any fixed *Q* and any given *w*. Thus $\int f(y - w, Q - w)\psi(w)dw$ is *K*-convex in *y* for any fixed *Q*, where $\psi(w)$ is the density function for *W*. For part (b), the proof is similar to the standard approach in Bertsekas (1995). Hence, we omit the details. □

In Lemma 8, our result in (*a*) is slightly different from the previous literature. In previous literature, it is presented as: for any given *Q*, if $f(y,Q)$ is *K*-convex in *y* and *W* is a random variable, then $\mathbb{E}[f(y - W, Q)]$ is *K*-convex. The difference is that in our model, both state variables are reduced by the same amount of random demand *W*. The result in (*b*) implies preservation properties of K-convexity. The next result is intuitive and we omit the proof.

LEMMA 9. *For any $t = 1,\ldots,T$, V_t, G_t, and V_{T+1} satisfy Property I.*

To characterize the optimal policy, we define two critical points using the cost functions. We can easily verify the continuity of the cost functions. It follows that the critical points are well defined as follows.

DEFINITION 4.

$$S_t(Q_t) = \min \arg \inf_{y_t} \{G_t(y_t, Q_t)\}; \tag{20}$$

$$s_t(Q_t) = \inf\{y_t | G_t(y_t, Q_t) \le K + G_t(S_t(Q_t), Q_t)\}. \tag{21}$$

We are now ready to present our main result in this section, that is, the optimal ordering policy is an unsold-commitment-dependent (s, S) policy.

THEOREM 10. *For any period t, $t = 1, 2, \ldots, T$,*

(a) V_t, G_t, and V_{T+1} are K-convex in y_t for any given Q_t.
(b) There exist $s_t(Q_t)$ and $S_t(Q_t)$ with $s_t(Q_t) \le S_t(Q_t)$, such that $y_t^*(x_t, Q_t) = S_t(Q_t)$ if $x_t < s_t(Q_t)$, and $y_t^*(x_t, Q_t) = x_t$ otherwise. Moreover, we have
  (i) $s_t(Q_t)$ and $S_t(Q_t)$ are constant in Q_t when $x_t \ge Q_t$, denoted by s_t and S_t;
  (ii) $s_t(Q_t) \le Q_t$, when $Q_t \ge s_t$; $S_t(Q_t) \le Q_t$, when $Q_t \ge S_t$;
  (iii) Let $S^{max} = \max\{S^M, S_1, S_2, \ldots\}$, then when $Q_t \ge S^{max} + (T - t)\bar{D}$, $s_t(Q_t)$ and $S_t(Q_t)$ are constant in Q_t.

PROOF. The K-convexity of G_t and V_t can be checked directly. The same can be said about the first part of part (b). See Bertsekas (1995) for details. The results that $s_t(Q_t)$ and $S_t(Q_t)$ are constant in Q_t when $x_t \ge Q_t$ come from Lemma 9 as follows. By Lemma 9, G_t is separable when $Q_t \le y_t$ and V_t is separable when $Q_t \le x_t$. Therefore, there exist $h_t^G(y_t)$, $l_t^G(Q_t)$, $h_t^V(x_t)$, and $l_t^V(Q_t)$, such that $G_t(y_t, Q_t) = h_t^G(y_t) + l_t^G(Q_t)$, $V_t(x_t, Q_t) = h_t^V(x_t) + l_t^V(Q_t)$. Then, we have $S_t(Q_t) = \min \arg \inf_{y_t}\{G_t(y_t, Q_t)\} = \min \arg \inf_{y_t}\{h_t^G(y_t) + l_t^G(Q_t)\} = \min \arg \inf_{y_t}\{h_t^G(y_t)\}$, which is a constant, denoted by S_t. Similarly, we have $s_t(Q_t) = \inf\{y_t | h_t^G(y_t) + l_t^G(Q_t) \le K + h_t^G(S_t(Q_t)) + l_t^G(Q_t)\} = \inf\{y_t | h_t^G(y_t) \le K + h_t^G(S_t)\}$, which is also a constant, denoted by s_t. That is, $s_t(Q_t)$ and $S_t(Q_t)$ are constant when $Q_t \le x_t$. □

For part (b) (ii), we show $s_t(Q_t) \le Q_t$ when $Q_t \ge s_t$ by contradiction. Suppose there exists $\bar{s}_t(Q_t) > Q_t \ge s_t$. Then, there exists $\bar{s}_t(Q_t) > z > Q_t \ge s_t$, such that $H_t(y_t^*(z, Q_t), Q_t) + K + \alpha V_{t+1}(y_t^*(z, Q_t), Q_t) < H_t(z, Q_t) + \alpha V_{t+1}(z, Q_t)$, where $y_t^*(z, Q_t)$ is the optimal order-up-to level. As shown by (i), $s_t(Q_t)$ and $S_t(Q_t)$ are constant in Q_t when $z \ge Q_t$. Therefore, $\bar{s}_t(Q_t) = s_t$ which contradicts $\bar{s}_t(Q_t) > s_t$. Hence, we have proven that $s_t(Q_t) \le Q_t$ when $Q_t \ge s_t$. $S_t(Q_t) \le Q_t$ when $Q_t \ge S_t$ can be shown in a similar way.

We finally show, given $Q_t \ge S^{max} + (T - t)\bar{D}$, $s_t(Q_t)$ and $S_t(Q_t)$ are constant in Q_t as follows. When $x_t \ge Q_t$ and $Q_t \ge S^{max} + (T - t)\bar{D}$, the optimal strategy is to stay because $s_t(Q_t) < S_t(Q_t) \le S^{max} \le Q_t$. Therefore, we only consider $x_t < Q_t$.

We now prove that when $x_t < Q_t$, for any $i \ge t$, $y_i^* < Q_i$, and $x_{i+1} < Q_{i+1}$ by induction as follows. We first prove that when $i = t$, $y_t^* \le Q_t$. Because $Q_t \ge S^{max} + (T - t)\bar{D} \ge S^{max} > S_t > s_t$, therefore by (ii), $s_t(Q_t) < S_t(Q_t) < Q_t$. When $x_t < Q_t$, the optimal order-up-to level $y_t^*(x_t, Q_t) = S_t(Q_t)$ if $x_t < s_t(Q_t)$, and $y_t^*(x_t, Q_t) = x_t$ otherwise. We have $y_t^* < Q_t$ because $x_t < Q_t$ and $s_t(Q_t) < S_t(Q_t) < Q_t$. $x_{t+1} = y_t^* - D_t < Q_t - D_t = Q_{t+1}$. Therefore the results hold for period t.

When we have $x_i < Q_i$, the optimal order-up-to level $y_i^*(x_i, Q_i) = S_i(Q_i)$ if $x_i < s_i(Q_i)$, and $y_i^*(x_i, Q_i) = x_i$ otherwise. We have $y_i^* \le Q_i$ because $x_i < Q_i$ and $s_i(Q_i) < S_i(Q_i) \le Q_i$ and $x_{i+1} = y_i^* - D_i \le Q_i - D_i = Q_{i+1}$. Therefore, the results hold for any $i \ge t$. For period T, since $Q_T \ge y_T, Q_{T+1} \ge x_{T+1}$, we have $G_T(y_T, Q_T) = (c_T - c_{T+1})y_T + L_T(y_T) + \alpha K + \alpha c_{T+1}\min\{y_T, 0\}$. When $x_t < Q_t, Q_t \ge S^{max} + (T-t)\bar{D}$, we have

$$\begin{aligned} G_t(x_t, Q_t) &= H_t(y_t) + \alpha \mathbb{E}[V_{t+1}(y_t - D_t, Q_t - D_t)] \\ &= H_t(y_t) + \alpha K \delta(y_{t+1}^*(y_t) - x_{t+1}) \\ &\quad + \alpha H_{t+1}(y_{t+1}^*(y_t)) + \alpha^2 \mathbb{E}[V_{t+2}(y_{t+1}^*(y_t) \\ &\quad - D_{t+1}, Q_{t+1} - D_t - D_{t+1}) | D_t, D_{t+1}] \\ &= \quad \cdots \\ &= \min_{y_t \geqslant x_t}\{H_t(y_t) + \alpha K \delta(y_{t+1}^*(y_t) - x_{t+1}) \\ &\quad + \alpha H_{t+1}(y_{t+1}^*(y_t)) + \cdots \\ &\quad + \alpha^{T-t} H_T(y_T^*(y_{T-1}^*(\ldots y_{t+1}^*(y_t)\ldots))) \\ &\quad + \alpha^{T+1-t} c_{T+1}(\min\{y_T, 0\} - \sum_{k=t}^{T}(\mathbb{E}D_k))\}, \end{aligned} \tag{22}$$

where $y_n^*, n = t + 1,\ldots,T$, is the optimal order up to level in period n, which is dependent on the previous inventory levels. Note that the last part of the above equation is not dependent on any Q levels. Therefore, $S_t(Q_t)$ and $s_t(Q_t)$ are constant numbers. □

Theorem 10 describes the optimal ordering policy, that is, order up to $S_t(Q_t)$ when the on-hand inventory

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is less than $s_t(Q_t)$, and order nothing otherwise. The optimal policy can be shown in a numerical study depicted in Figure 2. We consider stationary demand $D_t \sim N(10,1)$, per-unit cost $c_t = 10$, holding cost $h_t = 0.5$, backordering cost $b_t = 2, \forall t$, and setup cost $K = 30$. The planning horizon is 10 periods and we choose $t = 5$. Note that unlike the case $K = 0$, both reorder and order-up-to levels are dependent on Q_t.

For $Q_t \le x_t$, the problem is reduced to an inventory replenishment model without commitment. This is because the total minimum commitment has already been fulfilled. Therefore, $s_t(Q_t)$ and $S_t(Q_t)$ are the solutions to the T-period standard newsvendor problem without commitment. For $Q_t \ge (T-t)\bar{D} + S^{max}$, we can interpret it as the commitment is so large that we will not fulfill the commitment until the last period, thus we will only consider inventory (holding and backordering) costs and fixed setup cost. While we have proven that $s_t(Q_t)$ and $S_t(Q_t)$ are constant when Q_t is sufficiently low or sufficiently high, Figure 2 shows that $s_t(Q_t)$ and $S_t(Q_t)$ are neither constant nor necessarily monotone in Q_t.

4.2. Linearized Heuristic Policy

While we have shown in the previous subsection that the optimal policy is independent of x_t, it still depends on Q_t when $Q_t \in (x_t, (T-t)\bar{D} + S^{max})$. Hence, calculating $s_t(Q_t)$ and $S_t(Q_t)$ can still be time-consuming. To reduce the computational time, we propose a heuristic policy whereby $s_t(Q_t)$ and $S_t(Q_t)$ depend on Q_t linearly as shown in Figure 3. Specifically, we linearize $s_t(Q_t)$ and $S_t(Q_t)$ by connecting the values of s_t and S_t when they are constant in Q_t separately. We first obtain the optimal S_t and s_t when $x_t \ge Q_t$, then denote (S_t, S_t) as point 1 and (s_t, s_t) as point 3. Next, we compute the optimal S'_t and s'_t when

Figure 2 Reorder Level and Order-Up-To Level are not Monotone in Q_t

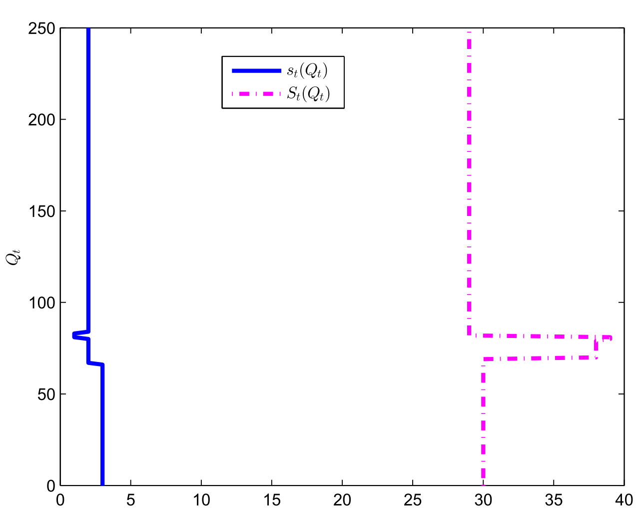
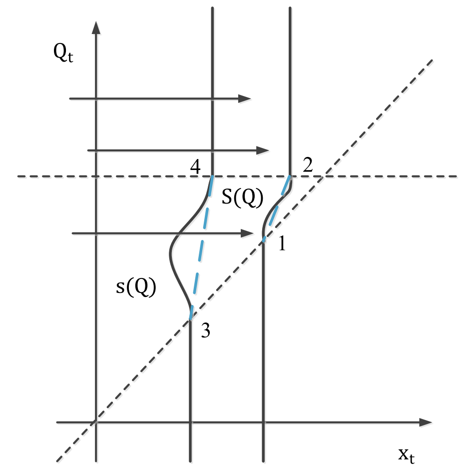


Figure 3 Linearized (*s*, *S*) Policy



$Q_t \ge (T-t)\bar{D} + S^{max}$, then denote $(S'_t, (T-t)\bar{D} + S^{max})$ as point 2 and $(s'_t, (T-t)\bar{D} + S^{max})$ as point 4. Finally, we connect point 1 and point 2 to get the linear function $S_t(Q_t) = \frac{(Q_t - S_t)(S'_t - S_t)}{(T-t)\bar{D} + S^{max} - S_t} + S_t$. Likewise, we connect point 3 and point 4 to get $s_t(Q_t) = \frac{(Q_t - s_t)(s'_t - s_t)}{(T-t)\bar{D} + S^{max} - s_t} + s_t$. This *linearized heuristic* works as follows: when $x_t \ge Q_t$ or $Q_t \ge (T-t)\bar{D} + S^{max}$, apply the optimal (*s*, *S*) policy; otherwise, follow the linearized $(s_t(Q_t), S_t(Q_t))$ policy.

In what follows, we numerically test the performance of the linearized heuristic vis-a-vis the optimal (*s*, *S*) policy in terms of expected total discounted cost. We do so for a variety of parameter settings: $T = 12$, $D_t \sim N(10,1), c_t \in \{1, 2, 4, 6, 8\}, h_t \in \{0.2, 0.4, 0.6, 0.8, 1\}, b_t \in \{1, 2, 3, 4\}, \forall t = 1, \ldots, T$, and $K \in \{5, 10, 15, 30, 45, 60\}$ for a total of 600 problem instances. We let $V_{opt}(x, Q)$ and $V_h(x, Q)$ be the total discounted cost for the optimal (*s*, *S*) policy and the linearized heuristic, respectively, when starting inventory is x and total commitment is Q. We further define, for each problem instance, $RE(x)$ as the relative error of the linearized heuristic with starting inventory x (worst-case over all realistic values of Q), that is,

$$RE(x) = \max_{Q \in [0, \bar{Q} - x]} \frac{V_h(x, Q) - V_{opt}(x, Q)}{V_{opt}(x, Q)} * 100\%$$

where $\bar{Q}$ equals the mean plus six standard deviations of total demand over all periods, that is, $\bar{Q} = T \cdot \mathbb{E}[D_t] + 6\sqrt{T} \cdot \sigma(D_t) = 10T + 6\sqrt{T}$. Tables 2, 3, and 4 summarize the results. In Table 2, we present for each K and each $x \in [0,100]$ the average and worst-case RE over 100 instances. We observe that RE follows an increasing trend with respect to x. This is because as starting inventory increases, the total cost decreases, which has a greater impact on the denominator of $RE(x)$ than its numerator. We also find that for a given x, RE increases as K

Table 2 Worst-Case and Average Relative Error (in %) as *K* Changes

K\x	0	10	20	30	40	50	60	70	80	90	100
Maximum RE											
5	1.59	3.03	1.93	3.38	2.34	3.72	2.62	3.91	2.72	3.40	2.59
10	2.85	3.97	4.25	4.11	5.05	5.18	4.62	5.14	4.52	4.80	5.22
15	4.16	4.83	5.25	6.47	5.86	6.62	6.97	6.80	7.15	7.69	8.11
30	7.89	8.94	9.79	10.89	11.28	11.44	12.98	14.13	15.00	15.64	16.20
45	11.38	13.00	14.61	15.89	15.67	17.30	19.27	20.65	21.71	22.46	23.24
60	14.42	15.79	16.62	18.35	20.20	22.07	24.33	25.65	27.35	28.29	28.92
Average RE											
5	0.46	0.61	0.55	0.69	0.62	0.76	0.68	0.82	0.73	0.88	0.82
10	1.04	1.41	1.30	1.58	1.46	1.81	1.55	1.87	1.62	1.91	1.88
15	1.44	1.95	1.80	2.22	2.08	2.48	2.23	2.70	2.44	2.77	2.93
30	3.03	3.57	3.79	4.05	4.35	4.66	4.87	5.04	5.23	5.69	6.14
45	4.54	5.07	5.40	5.99	6.35	6.75	7.08	7.61	8.11	8.74	8.87
60	5.70	6.68	7.08	7.74	8.00	8.62	9.50	9.90	10.79	11.35	11.20

Table 3 Worst-Case and Average Relative Error (in %) as *T* Changes

T\x	0	10	20	30	40	50	60	70	80	90	100
Maximum RE											
12	14.42	15.79	16.62	18.35	20.20	22.07	24.33	25.65	27.35	28.29	28.92
24	5.69	6.17	6.58	7.06	7.77	7.89	7.24	7.69	7.87	8.31	8.45
48	0.24	0.41	0.26	0.20	0.30	0.17	0.15	0.13	0.10	0.16	0.10
Average RE											
12	2.70	3.21	3.32	3.71	3.81	4.18	4.32	4.66	4.82	5.22	5.31
24	1.05	1.20	1.21	1.29	1.27	1.38	1.35	1.44	1.41	1.44	1.44
48	0.01	0.01	0.01	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00

Table 4 Worst-Case and Average Relative Error (in %) as Demand CV Changes

CV\x	0	10	20	30	40	50	60	70	80	90	100
Maximum RE											
0.1	14.42	15.79	16.62	18.35	20.20	22.07	24.33	25.65	27.35	28.29	28.92
0.2	13.14	14.19	14.98	15.19	16.94	18.70	20.64	22.34	23.84	24.62	24.71
0.3	11.85	12.43	13.19	12.83	14.74	16.54	18.16	19.95	20.77	21.72	21.10
0.4	10.13	10.81	11.45	10.84	12.68	14.37	16.18	17.64	18.54	19.51	18.19
Average RE											
0.1	2.70	3.21	3.32	3.71	3.81	4.18	4.32	4.66	4.82	5.22	5.31
0.2	2.07	2.34	2.52	2.70	2.86	3.04	3.23	3.47	3.76	4.04	4.00
0.3	1.74	1.89	2.06	2.19	2.33	2.46	2.64	2.84	3.17	3.39	3.22
0.4	1.46	1.59	1.72	1.84	1.98	2.11	2.27	2.49	2.82	3.01	2.83

increases, implying the increasing importance of finding the optimal (s, S) policy as fixed setup cost becomes larger. Moreover, average RE values appear to be reasonably low at 11.2% or less for all K and x values considered, while worst-case RE does not exceed 16.2% for $K \leq 30$. In Table 3, we vary the contract duration to $T = 24$ and $T = 48$ and run another 600 instances for each T and each x. Our findings reveal that for a given x, RE decreases as T increases. Finally, in Table 4, we alter the demand coefficient of variation (CV) from 0.1 to 0.2, 0.3, and 0.4, and again run another 600 instances for each CV and each x. Here, we fix the mean of the normally distributed demand at 10 and vary the standard deviation. Interestingly, for a given x, the RE values decrease as demand variability increases. This seems to suggest that increased demand variability harms both optimal and heuristic policies such that the percentage gap between them narrows. Overall, these results indicate that apart from worst-case performance when K and x are large and T is low, the linearized heuristic generally performs very well.

4.3. Hybrid Heuristic Policy

To further improve our linearized heuristic, we consider using the optimal (s, S) policy for the last β percentage of the contract duration and using the linearized heuristic for the first $1 - \beta$ percentage of the contract duration. For example, when $T = 12$ and

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β = 25%, we use the optimal (s, S) policy for the last βT = 3 periods. This requires negligible increase in computational time as it is relatively easy to find $s(Q)$ and $S(Q)$ for say a three-period problem. We call this policy the *hybrid heuristic* and conduct numerical tests to compare with the optimal policy and the linearized heuristic. Using the same problem parameters as subsection 4.2 and considering T = 12 and T = 24, Figure 4 shows that the hybrid heuristic provides some benefits when x is low. The improvement is most substantial for worst-case performance under the reasonable scenario of short contracts with low starting inventory.

5. Extensions

In this section, we examine two nontrivial extensions for which our proposed formulation allows analytical tractability of the optimal policies. These extensions demonstrate the general effectiveness of our approach. Before we present our main extensions, we briefly mention two straightforward extensions. The first is if we assume that backorders x_{T+1} do not need to be cleared, which is the case in Bassok and Anupindi (1997). The last period optimal decision becomes $y_{T+1} = \max\{x_{T+1}, Q_{T+1}\}$, whereas the state transitions in all other periods remain unchanged. It is then easy to show that the structures of the optimal policies remain the same. The second is if we assume that leftover inventory x_{T+1} can be salvaged at price $-h_{T+1}$ such that $c_{T+1} > -h_{T+1}$, where we recall that c_{T+1} is the unit price to clear the backorders in the last period. Since $c_{T+1} + h_{T+1} > 0$, our approach as well as the optimal policy structures also carry over to this scenario.

5.1. Extension I: Different Per-Unit Cost Beyond Commitment

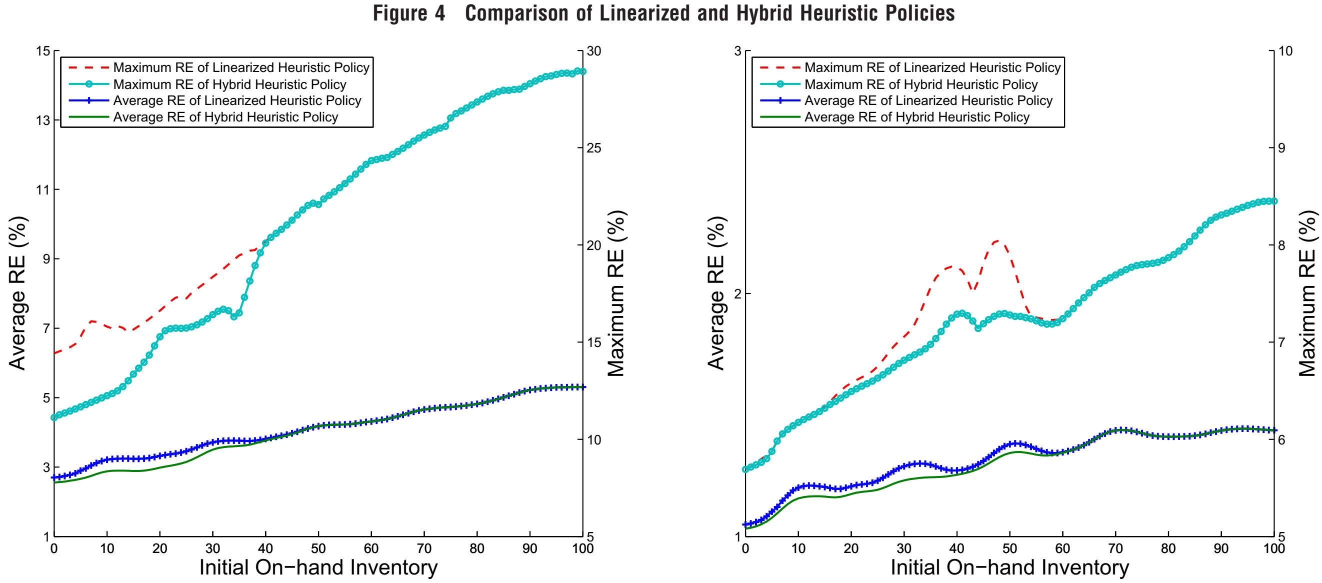
In this extension, we first consider the condition that the per-unit costs are stationary (i.e., $c_t = c$) and discounted (i.e., decreasing) to $c' < c$ when purchasing beyond the commitment. We further assume α = 1. This problem can be solved by calculating the extended costs in the last period. Specifically, we can let the unit cost be unchanged beyond the commitment but we deduct the cost difference in period $T+1$ as follows: $\overline{V}_{T+1}(x_{T+1}, Q_{T+1}) = K\delta(y_{T+1} - x_{T+1}) + cy_{T+1} - (c-c')(y_{T+1} - Q_{T+1}) = K\delta(y_{T+1} - x_{T+1}) + c'y_{T+1} + (c-c')Q_{T+1}$, where $y_{T+1} = \max\{x_{T+1}, Q_{T+1}, 0\}$.

For K = 0, since the cost is linear, it can be easily proven that Properties I and II are still satisfied, that is, this extension does not violate the structure of the value function. The optimal policy is of the same form as Theorem 7. When $K > 0$, for any $t = 1, \ldots, T$, it can be similarly proven that $\overline{V}_t$, $\overline{G}_t$, and $\overline{V}_{T+1}$ are K-convex in y_t (or x_t) for any given Q_t. Hence, the optimal policy is also an unsold-commitment-dependent (s, S) policy.

We can also extend the model to allow a discount factor $\alpha \le 1$ when $c' \ge c$. We have

$$\acute{V}_t(x_t, Q_t) = \min_{y_t \ge x_t} \Big\{ K\delta(y_t - x_t) + (c - \alpha c)y_t + \alpha c \mathbb{E} D_t + (c' - c)[(y_t - Q_t)^+ - (x_t - Q_t)^+] + L_t(y_t) + \alpha \mathbb{E} \acute{V}_{T+1}(y_t - D_t, Q_t - D_t) \Big\},$$

Figure 4 Comparison of Linearized and Hybrid Heuristic Policies



(a) $T = 12$ (b) $T = 24$

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for $t = 1, 2, \ldots, T$, with terminal condition

$$\acute{V}_{T+1} = K\delta(y_{T+1} - x_{T+1}) + cy_{T+1} + (c' - c)[(y_{T+1} - Q_{T+1})^{+} - (x_{T+1} - Q_{T+1})^{+}],$$

where $(y_t - Q_t)^{+} - (x_t - Q_t)^{+}$ in the purchase quantity in period t in excess of the MTC. Because $(y_t - Q_t)^{+} = (x_{t+1} - Q_{t+1})^{+}$, we can defer the cost accounting for $(c' - c)(x_{t+1} - Q_{t+1})^{+}$ to the next period $t + 1$. We obtain $\breve{V}_t(x_t, Q_t) = \min_{y_t \geq x_t} \{K\delta(y_t - x_t) + (c - \alpha c)y_t + \alpha c\mathbb{E}D_t + (c' - c)[\frac{(x_t - Q_t)^{+}}{\alpha} - (x_t - Q_t)^{+}] + L_t(y_t) + \alpha\mathbb{E}\breve{V}_{t+1}(y_t - D_t, Q_t - D_t)\}$, that is,

$$\breve{V}_t(x_t, Q_t) = \min_{y_t \geq x_t}\{K\delta(y_t - x_t) + (c - \alpha c)y_t + \alpha c\mathbb{E}D_t + (c' - c)\left(\frac{1}{\alpha} - 1\right)(x_t - Q_t)^{+} + L_t(y_t) + \alpha\mathbb{E}\breve{V}_{t+1}(y_t - D_t, Q_t - D_t)\}, \quad (23)$$

with terminal condition

$$\breve{V}_{T+1}(x_{T+1}, Q_{T+1}) = K\delta(y_{T+1} - x_{T+1}) + cy_{T+1} + (c' - c)[(y_{T+1} - Q_{T+1})^{+} + \left(\frac{1}{\alpha} - 1\right)(x_{T+1} - Q_{T+1})^{+}],$$

where $y_{T+1} = \max\{x_{T+1}, Q_{T+1}, 0\}$.

From Equation (23), we can show that the term $(c' - c)(\frac{1}{\alpha} - 1)(x_t - Q_t)^{+}$ is convex in Q_t when $c' \geq c$ and $\alpha \leq 1$. We can verify that Property I and Property II still hold for zero setup cost case and Property I holds for nonzero setup cost case. Hence, the structure of optimal policy in each case does not change under this setting.

5.2. Extension II: Nonzero Lead Time

In this extension, we take lead time into consideration to further generalize our basic model. Suppose there is a positive constant lead time L from order placement to arrival at the buyer. Then, the order placed in period t will arrive in period $t + L$. Let $T' = T + L$. It is obvious that the last ordering opportunity for the firm to fill demand occurring before the contract ends is in period $T' - L$. In other words, any orders placed after this period will not arrive before and at T'. Therefore, it is natural to fulfill all remaining commitment by T.

We use inventory position instead of on-hand inventory level as the state variable. Let z_t be the inventory position at the beginning of period t before ordering, and q_t be the order quantity in period t. Let $\bar{y}_t$ be the inventory position after ordering and x_t be the on-hand inventory at the beginning of period t after receiving q_{t-L}. Since inventory (holding and backordering) cost depends on on-hand inventory and not on inventory position, and on-hand inventory in period $t + L$ is no longer influenced by any ordering decision after period t, we can therefore assign the expected inventory (holding and backordering) cost in period $t + L$ to period t, $t = 1, 2, \ldots, T$. To that end, we define

$$\bar{L}_t(\bar{y}_t) = \alpha^{L}\mathbb{E}[L_{t+L}(x_{t+L})] = \alpha^{L}(h_{t+L} \cdot \mathbb{E}[\max\{x_{t+L} - D_{t+L}, 0\}] + b_{t+L} \cdot \mathbb{E}[\max\{D_{t+L} - x_{t+L}, 0\}]).$$

Note that $x_{t+L} = \bar{y}_t - D_{[t,t+L-1]}$, where $D_{[t,t+L-1]} = \sum_{n=t}^{t+L-1} D_n$. Hence, we have

$$\bar{L}_t(\bar{y}_t) = \alpha^{L}(h_{t+L} \cdot \mathbb{E}[\max\{\bar{y}_t - D_{[t,t+L]}, 0\}] + b_{t+L} \cdot \mathbb{E}[\max\{D_{[t,t+L]} - \bar{y}_t, 0\}]).$$

Similar to our basic model, we transform the state as follows: $Q_t = z_t + Q_t^{r}, Q_{t+1} = z_{t+1} + Q_{t+1}^{r} = \bar{y}_t - D_t + Q_t^{r} - (\bar{y}_t - z_t) = Q_t - D_t$. Under these settings, we can write the value function with z_t and Q_t as state variables. For $t = 1, 2, \ldots, T$,

$$\bar{V}_t(z_t, Q_t) = \min_{\bar{y}_t \geqslant z_t}\{K\delta(\bar{y}_t - z_t) + \bar{G}_t(\bar{y}_t, Q_t)\}, \quad (24)$$

where

$$\bar{G}_t(\bar{y}_t, Q_t) = \bar{H}_t(\bar{y}_t) + \alpha\mathbb{E}[\bar{V}_{t+1}(\bar{y}_t - D_t, Q_t - D_t)], \quad (25)$$

$$\bar{H}_t(\bar{y}_t) = (c_t - \alpha c_{t+1})\bar{y}_t + \alpha c_{t+1}\mathbb{E}D_t + \bar{L}_t(\bar{y}_t). \quad (26)$$

with boundary conditions

$$\bar{V}_{T+1}(z_{T+1}, Q_{T+1}) = K\delta(\bar{y}_{T+1} - z_{T+1}) + c_{T+1}\bar{y}_{T+1}, \quad (27)$$

$$\bar{y}_{T+1} = \max\{Q_{T+1}, z_{T+1}\}. \quad (28)$$

Please note that we have to fulfill the commitment in period $T + 1$ in order to receive the order by the end of the planning horizon. To find the structure of the optimal policy, we first consider the case of $K = 0$. For $t = T + 1$, we have

$$\bar{V}_{T+1}(z_{T+1}, Q_{T+1}) = \begin{cases} c_{T+1}z_{T+1}, & \text{if } z_{T+1} \geq Q_{T+1}; \\ c_{T+1}Q_{T+1},, & \text{if } z_{T+1} < Q_{T+1} \end{cases}.$$

For $t = T$,

$$\bar{V}_T(z_T, Q_T) = \min_{\bar{y}_T \geqslant z_T}\{\bar{H}_T(\bar{y}_T) + \alpha\mathbb{E}[\bar{V}_{T+1}(\bar{y}_T - D_T, Q_T - D_T)]\},$$

$$= \min\begin{cases} \min_{\bar{y}_T \geqslant z_T, \bar{y}_T \geq Q_T}\{\bar{gv}_T(\bar{y}_T)\}, \\ \min_{\bar{y}_T \geqslant z_T, \bar{y}_T < Q_T}\{\bar{gu}_T(\bar{y}_T)\} \\ \quad + \alpha c_{T+1}\mathbb{E}(Q_T - D_T), \end{cases}$$

where $\bar{gv}_T(\bar{y}) = \bar{H}_T(\bar{y}) + \alpha c_{T+1}\mathbb{E}(\bar{y} - D_T)$, and $\bar{gu}_T(\bar{y}) = \bar{H}_T(\bar{y})$. Hence, S_T and S_T^{M} under nonzero lead time case can be derived from the following critical fractile:

$$\Psi(S_T) = \frac{\alpha^L b_{T+L} - (c_T - \alpha c_{T+1}) - \alpha c_{T+1}\Phi(\bar{y})}{\alpha^L (h_{T+L} + b_{T+L})},$$

where $\Psi(\cdot)$ is the cumulative distribution function of $D_{[t,t+L]}$, and

$$\Psi(S_T^M) = \frac{\alpha^L b_{T+L} - (c_T - \alpha c_{T+1})}{\alpha^L (h_{T+L} + b_{T+L})}.$$

Since $\alpha c_{T+1}\Phi(\bar{y}) \geqslant 0$, $S_T \leqslant S_T^M$. We can use almost the same way to verify that Theorem 3, Lemma 4, and Lemma 6 still hold under this nonzero lead time case. Hence, the structure of the optimal policy is of the same type as the zero lead time case.

For the case when $K > 0$, it suffices to show that $\bar{L}_t(\bar{y}_t)$ is convex in $\bar{y}_t$, which can be easily verified from its definition. It then follows that the optimal policy is an unsold-commitment-dependent (*s*, *S*) policy. We omit the straightforward details.

6. Effect of Contract Terms on Buyer's Cost

Our analytical results in sections 4 and 5 allow us to numerically examine how the terms of the MTC contract affect the buyer's performance. Specifically, we study the effect of contract duration, lead time and total commitment on buyer's optimal cost.

6.1. Duration and Commitment

In this subsection, we consider six different contracts over the same planning horizon of 12 periods. We denote the six contracts as follows: 1*12 MTC represents one 12-period MTC contract, 2*6 MTC represents two consecutive six-period MTC contracts, and so on and so forth until 12*1 MTC represents a contract with periodic commitment. The last contract is an analytically simpler contract which has already been studied (Anupindi and Akella 1993; Henig et al. 1997; Moinzadeh and Nahmias 2000). We let total commitment in the 12 periods range from 0 to 240 with increments of 12 units. For example, if total commitment is 120 units, then 2*6 MTC refers to two six-period MTC contracts with commitments of 60 units each, while 3*4 MTC refers to three four-period MTC contracts with commitments of 40 units each. For our numerical study, we consider $D_t \sim N(10, 1)$, $c_t = 10, h_t = 0.5, b_t = 2, \forall t$, and $K \in \{0,15,30\}$. Figures 5(a)–5(c) show the behavior of the buyer's optimal cost as contract duration and total commitment change for each setup cost value considered.

Not surprisingly, the optimal cost is nonincreasing in contract duration and nondecreasing in total commitment, while the gap between the periodic commitment contract 12*1 MTC and any of the other contracts widens as fixed setup cost increases. This implies the importance of studying nonzero setup cost which is one of the elements that differentiates this study from the literature. Next, observe that the supplier and the buyer have conflicting duration preferences where periodic commitments are clearly too inflexible for the buyer. However, shifting from 12*1 MTC to 1*12 MTC, although good for the buyer, will result in too much instability for the supplier. Fortunately, this is not necessary as Figure 5 shows that when setup cost is sufficiently large and total commitment is not too large (e.g., less than sum of average demands), the cost gap can almost be closed by just shifting from 12*1 MTC to 6*2 MTC. This result provides a meaningful compromise between the two parties, that is, the supplier only needs to absorb a little bit more demand uncertainty in order to provide most of the cost savings that the buyer can get.

We now test the good performance of 6*2 MTC to see if a periodic commitment approximation can also achieve the same benefit. To this end, we consider a new six-period model with periodic commitment equal to that of each two-period contract in 6*2 MTC. Moreover, the demand in each new period is doubled (i.e., the mean is doubled but the standard deviation is multiplied by $\sqrt{2}$), and unit holding cost and unit backordering cost are also doubled because these costs are now incurred over twice the duration. Specifically, we examine a 6*1 MTC (i.e., periodic commitment) contract with $T = 6$, $D_t \sim N(20, \sqrt{2}), c_t = 10, h_t = 1, b_t = 4, \forall t$, and $K = 30$. Interestingly, we find that the gap between 6*2 MTC and 6*1 MTC ranges from 39.5% to 62.2% of the gap between 6*2 MTC and 12*1 MTC for total commitment $Q = 12,24,\ldots,144$ (see Figure 5(d)). The cost difference can be explained by the order frequency flexibility present in 6*2 MTC which 6*1 MTC does not have. This implies that substantial savings are left on the table when one approximates an MTC contract with a periodic commitment contract. We have hence found further evidence on the importance of studying MTC contracts, especially in the presence of nonzero setup cost.

6.2. Lead Time and Commitment

In this subsection, we conduct a numerical study to examine the trade-off between lead time and total commitment. We consider the following parameters: $T = 12$, $D_t \sim N(10, 1), c_t = 10, h_t = 0.5, b_t = 2, \forall t$, and $K = 30$. We further let lead time vary from 0 to 1, 2, and 3, and let total commitment range from 60 to 200. The results are shown in Figure 6. This study allows us to make trade-offs between committing to minimum total quantities to shorten the lead time vs. swallowing a larger lead time to reduce (or eliminate) the total

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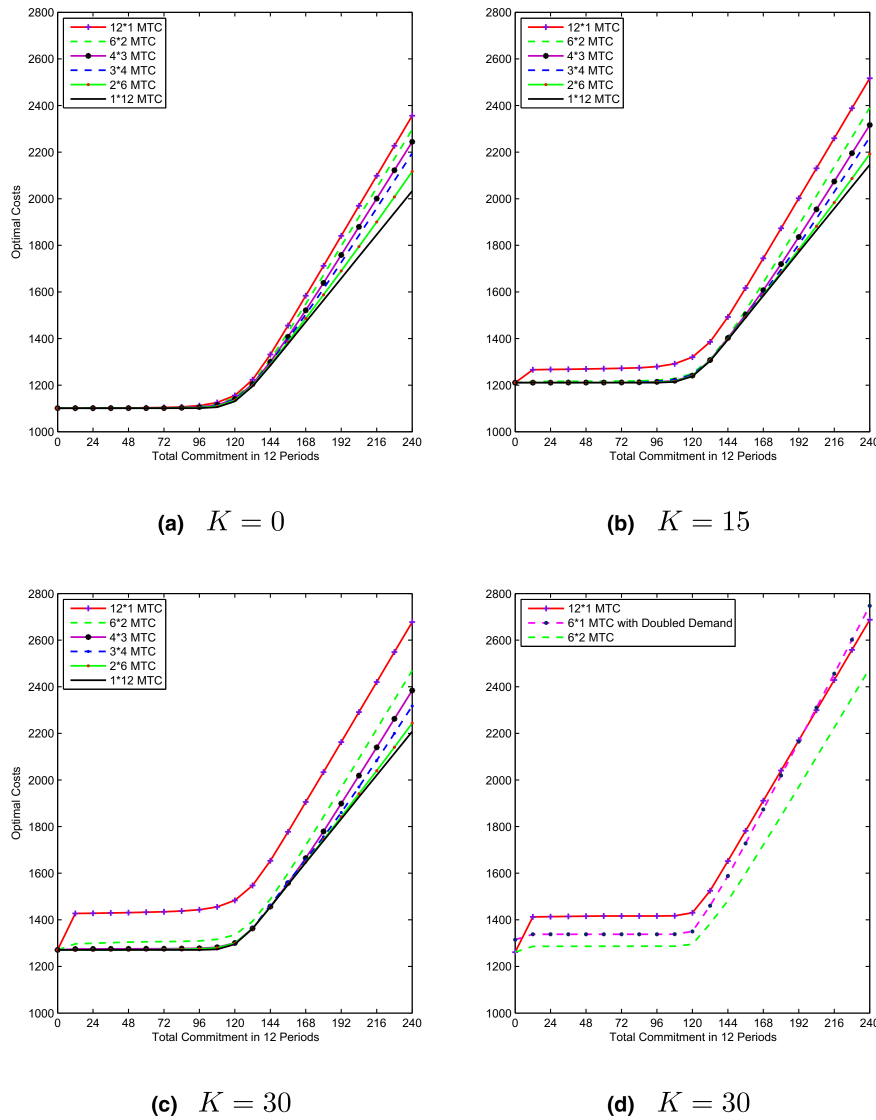
Figure 5 Optimal Costs with Different Contract Durations

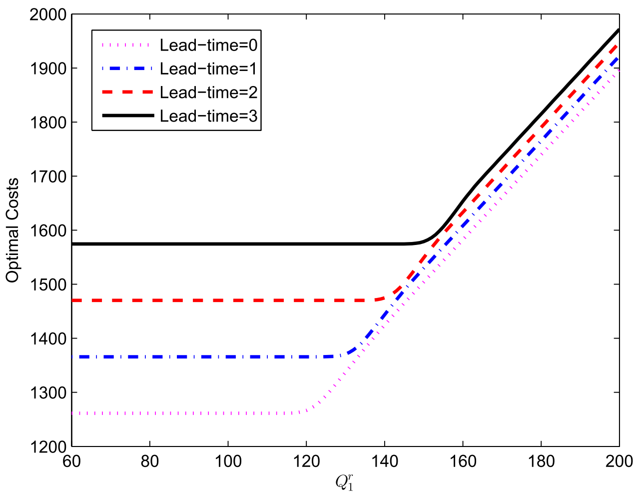


commitment. For example, when lead time is three periods and total commitment is 100 units, the buyer may negotiate to reduce lead time to 0. The figure shows that committing to $Q = 180$ will make the buyer worse off. However, committing to $Q = 140$ will be of value to both the buyer and the supplier. This confirms the observation that companies like Stanley Black & Decker Co. Ltd. may find it profitable to enter into MTC contracts in order to cut lead times, say from four weeks to one week (Zou 2012). However, the total commitment stipulated in the MTC contract must be carefully chosen. Figure 6 reveals that lead time reduction can lead to substantial savings if commitment is not too large. Moreover, a one-period reduction will have minimal savings if commitment exceeds some (lead time increasing) critical level.

7. Effect of Commitment Type on Supply Chain Profits

While it is clear that the buyer always prefers no commitment over MTC over periodic commitment, and the suppler always prefers the reverse, it is less clear which contract type is best for the entire supply chain. In this section, we consider a simple model to study this trade-off. First, we calculate expected buyer profit as unit revenue r multiplied by expected total demand $\sum_{t=1}^{T} \mathbb{E}[D_t]$, minus expected buyer cost as defined in the earlier sections. To do so, we generate $J = 1000$ sample paths $D_t^j, \forall t = 1, 2, \ldots, T, \forall j = 1, 2, \ldots, J$. For sample path j, we let $q_t^j, \forall t$ be the optimal buyer order quantities and compute the buyer's cost accordingly. We then average over all sample paths to obtain expected buyer cost.

Figure 6 Optimal Cost with Different *Q* When Lead Time Varies



Next, expected supplier profit can be obtained from $\frac{1}{J}\sum_{j=1}^{J}\sum_{t=1}^{T+1} c_t q_t^j$ less expected supplier cost. We model expected supplier cost as a function of both mean and variability of buyer orders, in two ways. The first model is

expected supplier cost

$$= \frac{1}{J}\sum_{j=1}^{J}\left\{\sum_{t=1}^{T+1} a \cdot q_t^j + f \cdot \sqrt{\frac{\sum_{t=1}^{T+1}(q_t^j - \bar{q}^j)^2}{T+1}}\right\}, \qquad (29)$$

where $\bar{q}^j = \frac{\sum_{t=1}^{T+1} q_t^j}{T+1}$, a is the unit production cost and f is the penalty for order variability.

For our second model, we define $T_q^j = \sum_{t=1}^{T+1} q_t^j$ as total buyer order quantity for sample path j. Then, we have

expected supplier cost

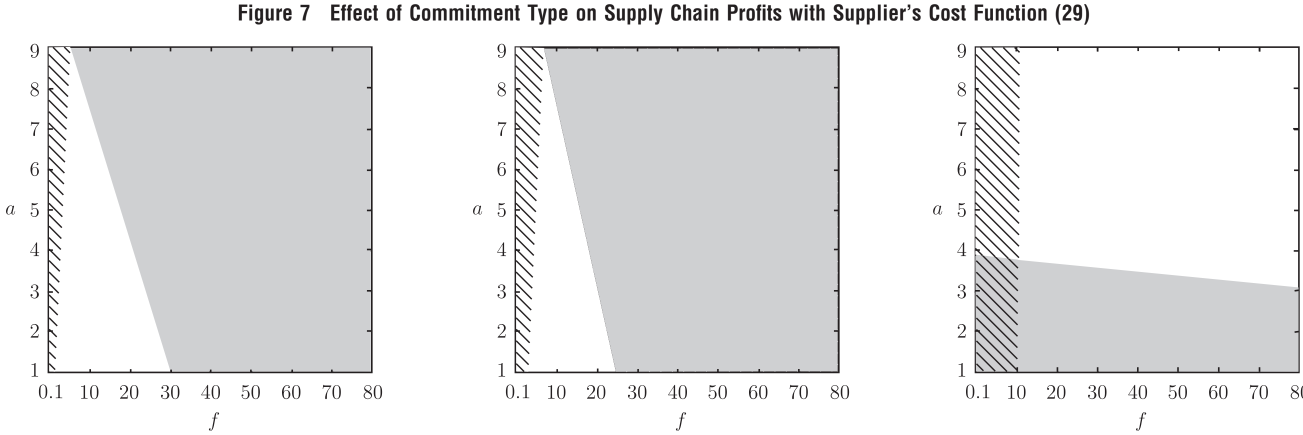
$$= a \cdot \frac{\sum_{j=1}^{J} T_q^j}{J} + f \cdot \sqrt{\frac{\sum_{j=1}^{J}(T_q^j - \bar{T}_q)^2}{J-1}}, \qquad (30)$$

where $\bar{T}_q = \frac{\sum_{j=1}^{J} T_q^j}{J}$, a is still the unit production cost and f is also the penalty for order variability, albeit a different type.

The first model considers the variation of orders across periods while the second considers the variation of total orders across all 1000 sample paths considered. Supply chain profit is then defined as the sum of buyer profit and supplier profit. Observe that the buyer's optimal order quantities depend on the commitment type and the total commitment but not on the unit revenue r. On the other hand, the expected supplier profit depend on buyer order quantities as well as cost parameters a and f. It follows that supply chain profit is also independent of r.

For our numerical study, we let $T = 12$, $D_t \sim N(10,1), c_t = 5, b_t = 2, h_t = 0.5, \forall t = 1,\ldots,T$, $K = 15$ and $x_1 = 0$. We consider commitment levels equal to 80%, 100% and 120% of average demand. That is, $Q_1 \in \{96,120,144\}$ for MTC contract and $MOQ \in \{8,10,12\}$ for periodic commitment contract where MOQ is the minimum order quantity in every period. For our two supplier cost models (29) and (30), we identify in Figures 7 and 8 the values of a and f for which MTC outperforms MOQ (lined region) and vice versa (unlined region), and for which MTC outperforms no commitment (shaded region) and vice versa (unshaded region). For cost model (29), Figure 7 shows that MTC outperforms MOQ when f is low (lined region), that is, order variability is less important. This region increases as commitment level increases. For MTC vs. no commitment, the pattern is more complicated as it depends on commitment level. For low and medium commitment levels, MTC outperforms no commitment when f and a are both high (shaded region), shifting more weight to supplier cost. For high commitment level, the region is less sensitive to f and MTC outperforms no commitment when a is low.

Figure 7 Effect of Commitment Type on Supply Chain Profits with Supplier's Cost Function (29)



(a) $Q_1 = 96, MOQ = 8, r = 12.5$ **(b)** $Q_1 = 120, MOQ = 10, r = 12.5$ **(c)** $Q_1 = 144, MOQ = 12, r = 12.5$

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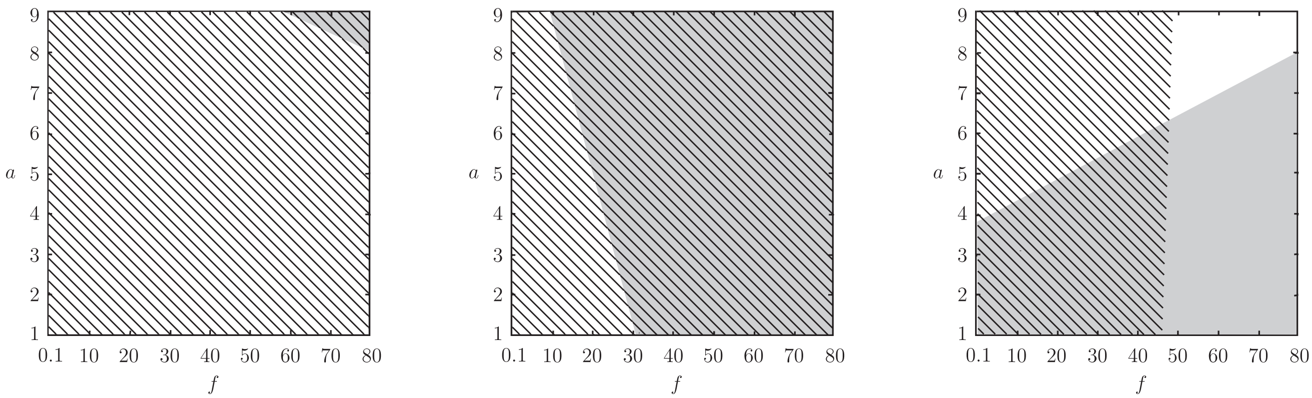
Figure 8 Effect of Commitment Type on Supply Chain Profits with Supplier's Cost Function (30)



(a) $Q_1 = 96, MOQ = 8, r = 12.5$ **(b)** $Q_1 = 120, MOQ = 10, r = 12.5$ **(c)** $Q_1 = 144, MOQ = 12, r = 12.5$

For cost model (30), Figure 8(c) shows a similar pattern as Figure 7(c), except for a much larger lined region where MTC outperforms MOQ as well as a much larger lined shaded region where MTC is the best of all three contract types. For medium commitment level, MTC is always better than MOQ for all a and f values considered. Between MTC and no commitment, MTC is preferred when f is high due to greater penalty for order variability. For low commitment level, MTC is again preferred over MOQ but commitment is too low that it brings more cost to the buyer than order variability reduction for the supplier. Hence, no commitment is preferred over MTC.

Overall, for either supplier cost model, the best commitment type depends on the commitment level, the supplier's unit production cost and order variability penalty. More interestingly, our study shows that MTC is a better contract for achieving stability of total orders, while MOQ is for production smoothing. This result clearly showcases the value of MTC contracts and under which scenarios they are most beneficial.

8. Conclusion

In this study, we examine a periodic review inventory system where there is a MTC on the replenishment quantities to be fulfilled by the end of a finite planning horizon. Unlike existing literature, we consider nonstationary per-unit cost, discount factor, and nonzero setup cost. Since the old formulations in existing literature cannot handle our more general setting, we develop a new formulation based on an intuitive yet powerful state transformation technique. By using unsold commitment instead of unbought commitment as the state variable, we discover that the value function evolves exogenously regardless of inventory decisions. This makes the analysis easier and we are able to fully characterize the optimal ordering policies. We demonstrate our approach by first revisiting the zero setup cost case, but with nonstationary per-unit cost and discount factor. We show that the optimal ordering policy is an unsold-commitment-dependent base-stock policy and provide a simpler proof for the optimality of the dual base-stock policy. We then analyze the case of nonzero setup cost, and prove for the first time that the optimal solution is an unsold-commitment-dependent (s, S) policy. We also design two easy-to-implement heuristic policies, which numerical tests show to perform very well. We also discuss two nontrivial extensions (per-unit cost different beyond commitment, and nonzero lead time) to demonstrate the general effectiveness of our approach. Finally, we use our results to examine how the buyer's optimal costs are affected by contract terms such as contract duration, lead time, and total commitment. We likewise compare total supply chain profits under periodic commitment, MTC and no commitment.

Our results are useful because they fully characterize the optimal ordering policies for MTC contracts under very general conditions. This implies the ability to make more comprehensive assessment and comparison among a larger set of MTC contracts. That is, the supplier and the buyer will have better information to design more effective supply contracts. Furthermore, with the results in this study, one will also be able to study how to allocate the fixed setup cost between the supplier and the buyer to better allocate risks and profits in the supply chain. Our results can also help in the design of a menu of contracts to offer to buyers with private information (e.g., demand). We leave these issues for future research.

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Note

[1]Yuan et al. (2013) consider a model in which permits for carbon emissions caused by production must be managed in each period through purchases or sales in the open market. In their paper, the commitment in each period is a decision variable but they do not have setup cost for production. They only have setup cost for changing the commitment. In our study, the total commitment is fixed but we have setup cost for production. The two models and their analyses are fundamentally different, and neither is a special case of the other.

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Supporting Information

Additional Supporting Information may be found in the online version of this article:

Appendix S1: Proof of Lemmas

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