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Optimal Discounting and Replenishment Policies for Perishable Products

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ABSTRACT

We consider a retailer, selling a perishable product with short shelf-life and uncertain demand, facing these key decisions: (a) whether to discount old(er) items, (b) how much discount to offer, and (c) what should be the replenishment policy. In order to better understand the impact of consumer behavior and shelf-life on these decisions, we consider four models. In Model A, the product has a shelf life of two periods and the retailer decides whether or not to offer a discount. The amount of discount is exogenous and assumed to be large enough so that all the customers prefer the old product to the new one when a discount is offered. Based on several numerical examples, we find that a *threshold* discounting policy, in which a discount is offered if and only if the inventory of old product is below a threshold, is optimal. In Model B, the retailer also decides how much discount to offer. Model C extends Model B and considers a *new pool* of customers who are willing to purchase from the retailer when a discount is offered. In both Models B and C, the product has a shelf-life of two periods while Model D explores what happens with longer shelf-life. We analyze and compare these models to present different managerial insights.

1. Introduction

Discounting policies can have important strategic implications for retailers. Recently, JC Penney, a well-known departmental store chain in the US instituted a "no sale" policy by having "everyday low pricing" (EDLP) and getting rid of sales through coupons/discounts. However, they suffered a backlash because consumers were not impressed with the move and had to go back to their old pricing strategy (Mourdoukoutas, 2013; Thau, 2013; Kapner, 2013). There are numerous other instances, e.g., perishable products such as milk and bread, where discounting is prevalent[1].

The discounting decision, especially in the context of promotional pricing vs. EDLP, has been studied in the past, largely in the marketing literature (e.g., see Ellickson and Misra (2008); Lal and Rao (1997), and the references contained therein). These research works largely analyze the problem at the "macro level" by considering it at the firm/store level with a wide range of products and typically hundreds of stock keeping units (SKUs). They also generally ignore operational aspects such as inventory issues and a limited shelf-life of the product. In this paper, we perform a "micro level" analysis by examining the discounting decision at the product level in which we consider both the replenishment and discounting policies in conjunction. While some research (see Section 2 for details) in operations management (OM) has looked at these policies together, we consider the operational factors and/or consumer behavior in more detail and granularity vis-a-vis these works[2]. In this regard, our paper complements research from both marketing and OM literatures, and *integrates important operational elements with the discounting decision by examining different types of consumer behavior in detail*.

We consider a retailer selling a perishable product with limited shelf-life under four different models. She reviews her inventory periodically and the lead time for getting the product is assumed to be zero. In Model A, the base model, we focus on the decision of whether a discount should be offered or not. The shelf-life of the product is two periods. The amount of discount is exogenous and assumed to be large enough so that when the old product (of age one) is discounted, all the customers prefer it to the new one (of age zero). Model B also considers the decision of how much to discount. It does so by modeling consumer behavior in more detail. The fraction of customers preferring the old product to the new one depends on the amount of discount that is offered. Model C further extends Model B by considering a *new pool* of customers who are willing to purchase from

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[1] Although apparel products (and also consumer electronics) do not physically decay over time, they do become obsolete quickly, are often discounted, and can be viewed as *perishable* products, especially in *fast fashion* scenarios.

[2] For instance, we consider both FIFO and LIFO inventory systems, based on the discounting decision in Model A while many research papers explicitly assume either FIFO or LIFO and that it can be *enforced* on the customers.

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the retailer when a discount is offered. The size of this pool depends both on the size of original customers and the amount of discount. Models A-C can be considered as incorporating different kinds of customers: in Model A, customers only look at whether the discount is offered; in model B, the *level* of discount is also important to them; while Model C considers additional new customers from discounting. While all these models assume a shelf-life of two periods, Model D extends Model A in another dimension by considering longer product shelf-life. Thus, we examine and compare the four models (Models A-D) to better understand how consumer behavior and product shelf-life affect the retailer's discounting and replenishment decisions and her profits.

The rest of the paper is organized as follows. In Section 2, we discuss the related literature. In Section 3, we describe the four different models mentioned above. Section 4 analyzes the models and presents different observations based on our numerical analyses. Finally, we conclude in Section 5. Throughout the paper, we use the terms “increasing” and “decreasing” in the weak sense.

2. Literature survey

The research in this paper is mainly related to the literature on perishable inventory management, joint optimization of pricing and inventory, and modeling of customer behavior for making optimal pricing and inventory decisions.

First, we consider the research on perishable inventory management. Nahmias (1975), a pioneering work in this area, finds the optimal inventory policy for a product with a multi-period shelf life. He shows that the decision of when to order depends only on the total number of old units but how much to order does depend on the distribution of old units across different ages. Pierskalla (1969) uses dynamic programing to analyze inventory issues when perishable products have random lifetimes. Pierskalla and Roach (1972) find optimal issuing policies when there is a limited supply of perishable products. When demand is fully back-ordered they show that first-in-first-out (FIFO) policies are optimal. Nahmias (1982, 2011) provides a review of the perishable goods supply chain literature with models that consider different features such as e.g., random vs. deterministic lifetime, stochastic vs. deterministic demand etc.). Deniz et al. (2010) consider heuristics for inventory issuing and replenishment policies when perishable products are substitutable, while Haijema (2014) also analyzes optimal disposal policies and finds how much value is added by these policies. We model the impact of pricing/discounting decisions on demand and replenishment decisions, which none of the above articles consider.

Second, we consider research involving joint optimization of pricing and inventory decisions. Petruzzi and Dada (1999) presents a review of different price-setting newsvendor models. Zabel (1970) considers a seller with a finite horizon, and shows that the optimal price is decreasing both in time and in the on-hand inventory, under certain conditions. Under a base stock list price (BSLP) policy, if the inventory is below a threshold, then it is brought up to this level and the optimal price is charged; otherwise, there is no production and the product is sold at a discount (that depends on the amount of inventory). Zabel (1970) shows that the BSLP policy is optimal under certain conditions. Thowsen (1975) also considers back-ordering and shows that BSLP policy can still be optimal. Federgruen and Heching (1999) find that BSLP is optimal even under infinite horizon provided the seller has price flexibility, i.e., prices are allowed to increase over time. Li et al. (2009) also analyze joint pricing and inventory control for perishable products but they do not consider how customers choose between new and old products. Chen and Smichi-Levi (2004) consider ordering costs in the above model and show that the (s, S, p) policy, which is similar to the standard (s, S) policy but the price charged depends on the on-hand inventory, is optimal. They also conclude that, unlike in the BSLP, the price function is not necessarily decreasing in the inventory level. Hopp and Xu (2006) consider a single period problem with dynamic pricing. The customer arrival rate is stochastic, and is assumed to follow a geometric Brownian motion. They find the optimal pricing policy and order quantity, and show that pricing dynamically can result in significant higher profits.

Some research works, which include Rajan et al. (1992) and Cohen (2006), have analyzed pricing and/or replenishment policies in the context of physically decaying perishable products (e.g., agricultural produce). Elmaghraby and Keskinocak (2003) provide a detailed review of dynamic pricing in different scenarios involving inventory issues (e.g., strategic vs. myopic customers, replenishment vs. non-replenishment of inventory etc.). Some recent research works in this area include Avinadav et al. (2013), Herbon et al. (2014), Chew et al. (2014), van Donselaar et al. (2016), Pauls-Worm et al. (2016), Chen (2017), and Feng et al. (2017). They do not model the discounting and/or replenishment decisions in the manner in which we do in this paper. In particular, *we consider lost sales and explicitly model the shift of customers from new to old products due to discounting of the latter by the retailer*.

Finally, we consider work that involves modeling of customer behavior for making better pricing and inventory decisions. Ferguson and Koenigsberg (2007) consider a two-period model with pricing and inventory optimization of new and old products. A different version of the problem, in which the new and old products cannot be simultaneously sold, is analyzed by Li et al. (2012). They suggest that there is a need for models aimed at operational decisions, which consider simultaneous sales of both new and old products. This paper considers this aspect. Our work comes close to Sainathan (2013) who consider pricing and inventory optimization and use a vertical differentiation model for customers having different perceptions of quality for new and old products. However, we specifically study how discounting affects aggregate demand from customers and focus on the structural properties, which we derive from several numerical examples, involving the optimal decisions. We also derive some insights on how the optimization problem for a problem with more than two-period shelf-life can be simplified, and find that the profit reduction from implementing a simpler policy is often insignificant (see Model D in Section 4.4).

In summary, some key aspects, which have not been considered together before to the best of our knowledge, make this paper unique: (i) focus on discounting decisions and how they affect and are affected by consumer behavior, and (ii) analyze how discounting decisions affect and are affected by inventory replenishment decisions.

3. General model

We consider a retailer, who follows periodic inventory replenishment, selling a perishable product with a shelf life of n periods over a finite horizon of $T \gg n$ periods. Therefore, in any period, she sells n versions of the product with ages $0, 1, \ldots, n-1$, where age 0 refers to the new units, while the others refer to older units of various ages. Unsold units of age i at the end of a period get transferred to the next period as units of age $i+1$, for $i = 0, 1, \ldots, n-2$. Unsold units of age $n-1$ are discarded at the end of a period at zero salvage value, without any loss of generality. The retailer can only procure new units and the cost of procurement per unit is c.

We count time backwards so that period T denotes the beginning of the horizon whereas period 0 refers to the end of the horizon. At the beginning of period t ($t = T, T-1, \ldots, 1$), the retailer reviews the inventory levels of old items, denoted by $\mathbf{s_t} = [s_{t,1}, s_{t,2}, \ldots, s_{t,n-1}]'$. Subsequently, she decides the quantity q_t of new units to order and whether or not to offer a discount $x_t > 0$ for the older units. For simplicity, we assume that the procurement lead time is zero and the retailer either discounts all available units of a certain age or none of them. The retailer makes these decisions in order to maximize her expected profit-to-go, i.e. the total expected profit from period t until the end of the horizon. We denote the optimal expected profit-to-go function in period t by $\pi_t(\mathbf{s}_t)$. If the discount is not offered for units of age $i(i = 0, 1, \ldots, n-1)$, then each of them is sold at an exogenous unit price p. We further suppose that no discounts will be given for new units.

We examine this problem under different settings in Sections 3.1–3.4 below. Models A-C, which are considered in Sections 3.1–3.3 respectively, incorporate different kinds of customers: in Model A, customers only look at whether the discount is offered; in model B, the *level* of discount is also important to them; while Model C considers additional new customers from discounting. Model D, in Section 3.4, extends Model A in another dimension by considering longer product shelf-life. Thus, we examine and compare the four models (Models A-D) to better understand how consumer behavior and product shelf-life affect the retailer's discounting and replenishment decisions and her profits. For all these models, we assume that when no discount is offered, customers would discern and prefer to purchase the most recent (new) units. Also, we assume that there is a base demand D_t, faced by the retailer when none of the units are discounted, in period t with pdf ϕ and cdf Φ; any unsatisfied demand is lost. Next, we describe each model in detail.

3.1. Model A

Here, we assume that (a) the shelf life is two periods (n=2), and (b) the discount $x_t = \delta > 0, \forall\, t$ is exogenous and large enough so that all customers from the base demand D_t will strictly prefer the older units (of age 1) over the new units. For the sake of brevity, we use s_t to denote the scalar inventory level of older units at the beginning of period t. We let $y_t \in \{0, 1\}$ be the binary variable that indicates whether or not the retailer discounts the older units in period t. Then, the optimal expected profit-to-go function can be written recursively as follows.

$$\begin{aligned}\pi_t(s_t) = \max_{q_t \ge 0, y_t \in \{0,1\}} &\{pE[\min(q_t + s_t, D_t)] - \delta y_t E[\min(s_t, D_t)] \\ &-cq_t + E[\pi_{t-1}((q_t - (D_t - s_t y_t)^+)^+)]\}, \forall\, t > 0;\\ \pi_0(.) = 0.&\end{aligned} \tag{1}$$

The first term inside the maximization accounts for the expected revenue in period t without discounting; the second term pertains to the expected loss in revenue in period t due to discounting; the third term refers to the procurement cost; and the fourth term is the expectation over D_t of the optimal expected profit-to-go function in period $t-1$.

3.2. Model B

This model is a generalization of Model A in which the discount $x_t \ge 0$ is also a decision variable. This fundamentally alters the manner by which base demand affects the demand for units of each age. In particular, we let $\alpha(x_t)$ be the fraction of base demand D_t that prefers the old units to new units if the discount offered is x_t. The rest of the base demand prefers new units to old units. Because q_t and s_t are the number of new units and the number of old units, respectively, we can characterize the sales of new and old units as $\min(q_t, (1-\alpha(x_t))D_t + (\alpha(x_t)D_t - s_t)^+)$ and $\min(s_t, \alpha(x_t)D_t + ((1-\alpha(x_t))D_t - q_t)^+)$, respectively. Note that the sum of these two sales quantities is $\min(q_t + s_t, D_t)$, which is also the total units sold in Model A. One advantage of this model is that it does not assume any priority in demand fulfillment among customers, hence we do not need to model the sequence in which the customers arrive at the retailer. The only assumption we make on $\alpha(x_t)$ is that it is nondecreasing in x_t and $\alpha(0) = 0$. We can observe that Model A is a special case in which $\alpha(x_t)$ is a step function, equal to 0 for all $x_t < \delta$ and equal to 1 for all $x_t \ge \delta$. Similar to Model A, the optimal expected profit-to-go function can be written as follows.

$$\begin{aligned}\pi_t(s_t) = \max_{q_t \ge 0, x_t \in [0,p]} &\{pE[\min(q_t + s_t, D_t)] - x_t E[\min(s_t, \alpha(x_t) \\ &D_t + ((1-\alpha(x_t))D_t - q_t)^+)] \\ &-cq_t + E[\pi_{t-1}((q_t - (1-\alpha(x_t))D_t - (\alpha(x_t)D_t - s_t)^+)^+)]\}, \forall\, t > 0;\\ \pi_0(.) = 0.&\end{aligned} \tag{2}$$

The equations in (2) have similar interpretation as those in (1). Alternatively, we can also write the optimal expected profit-to-go function as follows.

$$\begin{aligned}\pi_t(s_t) = \max_{q_t \ge 0, x_t \in [0,p]} &\{pE[\min(q_t, (1-\alpha(x_t))D_t + (\alpha(x_t)D_t - s_t)^+)] + (p - x_t) \\ &E[\min(s_t, \alpha(x_t)D_t + ((1-\alpha(x_t))D_t - q_t)^+)] \\ &-cq_t + E[\pi_{t-1}((q_t - (1-\alpha(x_t))D_t - (\alpha(x_t)D_t - s_t)^+)^+)]\}, \forall\, t > 0;\\ \pi_0(.) = 0.&\end{aligned} \tag{3}$$

The first term inside the maximization accounts for the expected revenue from new units in period t; the second term pertains to the expected revenue from old units in period t; the third term refers to the procurement cost; and the fourth term is the expectation over D_t of the optimal expected profit-to-go function in period $t-1$. While the above two formulations are equivalent, the latter helps us in our formulation in Model C.

3.3. Model C

Here, we further generalize Model B so that there is a new pool of customers (in addition to base demand D_t) who are attracted to the retailer mainly because of the discount. The size of this pool is increasing in the discount x_t. It also increases in the number of customers from the base demand who are attracted to the old units because of the discount, $\alpha(x_t)D_t$, and subsequently advertise about it through direct word-of-mouth to this pool. Specifically, we let the size of this pool be $bx_t \cdot \alpha(x_t)D_t$ where $b \ge 0$ is the indicator of the effectiveness of the word-of-mouth advertising. However, these customers will not buy the new units in the event that the old units are out of stock. We define $f(\gamma, b, x_t, D_t, s_t)$ to be the *effective sales* from this new pool, which is the number of customers from this pool whose demand is satisfied. This quantity will depend on the sequence in which the customers in this pool and the customers from the base demand who prefer old units arrive. We model this aspect through a parameter $\gamma \in [0, 1]$ which denotes the fraction of customers from this pool who arrive before the base demand (equivalently, the fraction of such customers who arrive after the base demand is $1-\gamma$). We let Model C1 ($\gamma = 1$) and Model C2 ($\gamma = 0$) be the following extreme cases, respectively: (1) all customers from the new pool arrive first, and (2) all customer from the new pool arrive last. Then, it can be shown that $f(\gamma, b, x_t, D_t, s_t) = \min(\gamma b x_t \alpha(x_t)D_t, s_t) + \min((1-\gamma)bx_t\alpha(x_t)D_t, (s_t - \gamma b x_t \alpha(x_t)D_t - \alpha(x_t)D_t)^+)$. The first (second) term corresponds to sales of old units from the customers in the new pool who arrive before (after) the base demand. The number of customers who prefer new units to old units is unchanged from Model B and remains equal to $(1-\alpha(x_t))D_t$. Note that when b=0, $f(.) = 0$ and we obtain Model B as a special case. Next, similar to (3) in Section 3.2, we formulate the optimal expected profit-to-go as follows.

$$\pi_t(s_t) = \max_{q_t \ge 0, x_t \in [0,p]} \{pE[\min(q_t, (1-\alpha(x_t))D_t + (\alpha(x_t)D_t + f(\gamma, b, x_t, D_t, s_t) - s_t)^+)] + (p - x_t)E[\min(s_t, \alpha(x_t)D_t + f(\gamma, b, x_t, D_t, s_t) + ((1-\alpha(x_t))D_t - q_t)^+)] - cq_t + E[\pi_{t-1}((q_t - (1-\alpha(x_t))D_t - (\alpha(x_t)D_t + f(\gamma, b, x_t, D_t, s_t) - s_t)^+)^+)]\}, \forall\, t > 0;$$
$$\pi_0(.) = 0. \tag{4}$$

In (4), because $f(\gamma, b, x_t, D_t, s_t) \le s_t$, the term $(\alpha(x_t)D_t + f(\gamma, b, x_t, D_t, s_t) - s_t)^+$ is the number of customers from the base demand who prefer the old units but are unable to purchase them.

3.4. Model D

This model generalizes Model A in a different direction so that the product has a general shelf life $n \ge 2$. In order to study better the effect of shelf life on the retailer's optimal decisions and profits, we make this generalization from Model A (instead of Model B or Model C). Recall that $x_t = \delta > 0$ is the exogenous discount and the state vector $\mathbf{s_t}$ refers to the inventory levels of units of various ages at the beginning of period t. The retailer has to decide not only the order quantity q_t, but also whether or not to discount units of each age i ($i = 1, 2, \ldots, n-1$). We denote the discounting decisions by $\mathbf{y_t} \in \{0, 1\}^{n-1}$. Next, we describe how the demand for units of different ages are related to the base demand and the discounting decisions. Customers make purchase decisions, first based on price and then on the age of the units. Specifically, if there are units of multiple ages that are discounted, customers will prefer the most recent of these discounted units. For the purpose of simplifying the formulation, we denote $\mathbf{M_t} = [M_{t,0}, M_{t,1}, \ldots, M_{t,n-2}]'$ as the demand vector for units of ages 0 to $n-2$. We now characterize $\mathbf{M_t}$ as a function of D_t, $\mathbf{s_t}$, $\mathbf{y_t}$, and q_t below.

$$M_{t,0} = \left(D_t - \sum_{j=1}^{n-1} s_{t,j} y_{t,j}\right)^+$$
$$M_{t,i} = \left(D_t - \sum_{j=1}^{i-1} s_{t,j} y_{t,j}\right)^+ y_{t,i} + \left(D_t - q_t - \sum_{j=1}^{i-1} s_{t,j} - \sum_{j=i+1}^{n-1} s_{t,j} y_{t,j}\right)^+ (1 - y_{t,i}), \forall\, i = 1, \ldots, n-2$$

The demand for new units $M_{t,0}$ comes from the base demand in excess of all discounted units, regardless of age. For older units of age i, the demand $M_{t,i}$ depends on whether these units are discounted or not. If they are discounted (i.e. $y_{t,i} = 1$), then it is the base demand in excess of all newer units on discount. Otherwise, it is the base demand in excess of all newer units (whether discounted or not) and all older discounted units. We can then write the formulation of the optimal expected profit-to-go as follows.

$$\pi_t(\mathbf{s_t}) = \max_{q_t \ge 0, \mathbf{y_t} \in \{0,1\}^{n-1}} \left\{ pE\left[\min\left(q_t + \sum_{i=1}^{n-1} s_{t,i}, D_t\right)\right] - \delta E\left[\min\left(\sum_{i=1}^{n-1} s_{t,i} y_{t,i}, D_t\right)\right] - cq_t + E[\pi_{t-1}(([q_t, s_{t,1}, \ldots, s_{t,n-2}]' - \mathbf{M_t})^+)]\right\}, \forall\, t > 0;$$
$$\pi_0(.) = 0. \tag{5}$$

The first three terms inside the maximization in (5) are similar to those in (1). The argument in the fourth term captures the state transformation of the inventory levels from period t to period $t-1$.

4. Analysis

We use the framework of dynamic programming to find the optimal decisions[3] for the retailer in any period. We first start with the analysis and results of Model A.

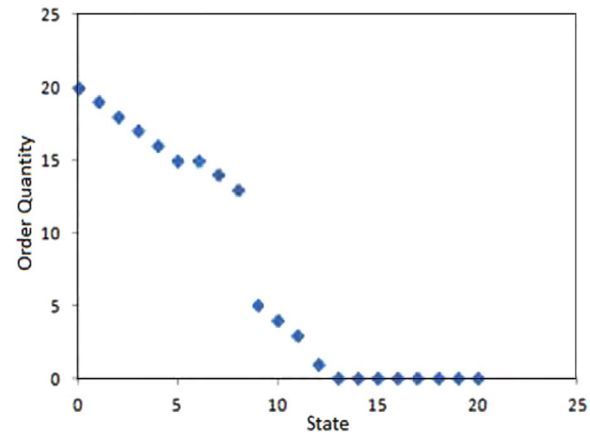


Fig. 1. Optimal order quantity with $D_t \sim U[0, 25]$.

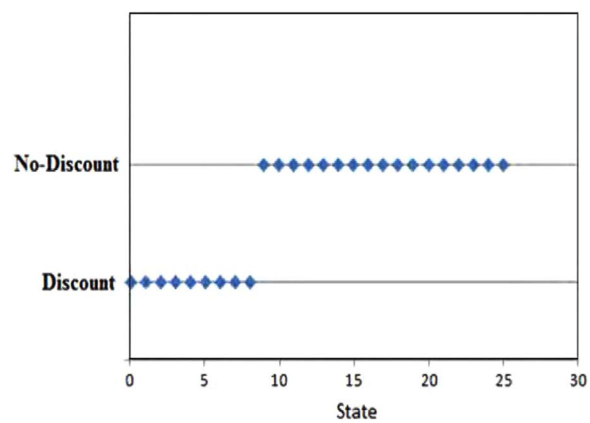


Fig. 2. Optimal discounting policy with $D_t \sim U[0, 25]$.

4.1. Model A

Model A is the base model with two-period shelf life (i.e. n=2) and discount $x_t = \delta > 0$ large enough to attract all customers to prefer the older units over new units. We examine this model for various discrete demand distributions such as uniform, binomial and negative-binomial[4]. We report our findings below.

4.1.1. Structure of optimal policy

Figs. 1, 2 and 3 show the effect of the inventory of old units on the optimal order quantity, discounting policy, and profit for uniformly distributed demand.[5] First, we find that the optimal order quantity is decreasing in inventory of old units. This is consistent with standard inventory literature as more leftover inventory means less need to procure new units. Second, the optimal discounting policy is a threshold policy such that a discount is offered if the inventory of old units falls below the threshold and no discount is offered otherwise. At first, this result seems counter-intuitive because more inventory usually suggests higher propensity to give a discount. However, the opposite is true here because the discount serves a different purpose which is to redirect customers from new units to old units, thereby freeing up the new units for future demand and not wasting the soon-to-perish old units. It follows that discount is more likely to be given when there are more new units, which from the first result occurs when there are less

[3] We find these values numerically in cases involving specific examples; in such cases, these values are *approximately* optimal even though we refer to them as optimal values for the sake of conciseness. We thank an anonymous reviewer for indicating this point.

[4] We use these distributions because they are discrete valued and computationally simple to handle.

[5] We observe similar results for binomial and negative-binomial distributions.

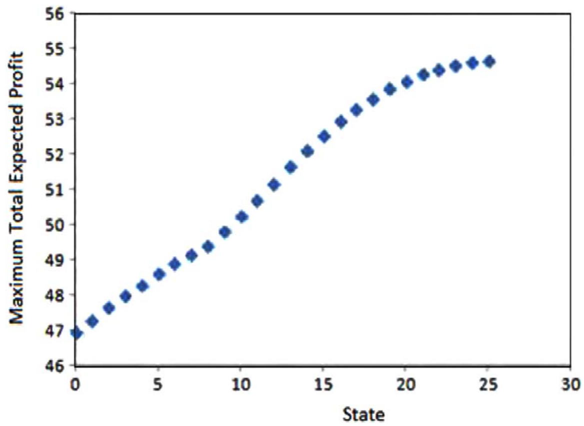


Fig. 3. Optimal profit with $D_t \sim U[0, 25]$.

old units. As seen in Fig. 1, there is a also significant decrease in optimal order quantity at the threshold level[6]. Finally, we observe that optimal profit is an increasing piecewise concave function (with two pieces) in the inventory of old units. This makes sense as more inventory means more resources[7] and less need to procure new units, hence higher profits. Concavity implies diminishing marginal value while the piecewise nature is due to the optimal discounting policy being a threshold policy.

4.1.2. Sensitivity analysis

We now examine the effect of demand uncertainty (i.e. standard deviation) on the structure of the optimal policy. We consider the binomial and negative-binomial distributions, holding the mean constant while changing the parameters to create low, medium and high levels of demand standard deviation. Figs. 4a, 5a and 6a display the results for the binomial distribution while Figs. 4b, 5b and 6b show the ones for the negative-binomial distribution.

Figs. 4a and 4b show that the behavior of the optimal order quantity as a function of inventory of old units is unaffected by demand standard deviation. However, the difference in optimal order quantities before and after the threshold increases in demand standard deviation. Moreover, we observe that for a given inventory of old units, the optimal order quantity is increasing in demand standard deviation.

Next, Figs. 5a and 5b tell us that the threshold discounting policy remains optimal regardless of demand standard deviation. However, we find that the threshold level is a increasing function of demand standard deviation. This means that as demand becomes more uncertain, the retailer is more likely to offer a discount at any given inventory level of old units.

Finally, Figs. 6a and 6b suggest that the optimal profit is still an increasing piecewise concave function whatever the demand standard deviation. However, for a given inventory of old units, the optimal profit is decreasing in demand standard deviation. This implies that more demand uncertainty harms the retailer more.

We also study the unit procurement cost c and discount δ on the optimal discounting policy. Fig. 7 summarizes the values of c and δ for which the retailer will give a discount at some inventory level of old units and the values for which the retailer never discounts. These regions are denoted in the figure as D and ND. Interestingly, we find that for a given δ, there exist a lower bound and an upper bound on c for the region D. This is because when the unit procurement cost is too high, offering the discount may lead to negative profit. On the other hand, when unit procurement cost is too low, the profit level is generally high and offering discount may lead to large profit losses. Furthermore, we find that as δ increases, the lower bound increases while the upper bound decreases. This makes sense because the higher the discount needed to attract all customers to buy old units, the less likely it is for the retailer to offer the discount. Finally, we observe that even in the case where $1 - c < \delta$, there exists an inventory level of old units in which it is optimal for the retailer to offer the discount. *Hence, even if the profit margin is less than the discount, the retail may still offer the discount.*

Based on the results presented above, we summarize the following observations about Model A:

Observation 1. The optimal discounting policy is a threshold policy. The optimal order quantity decreases in the inventory of old product and decreases significantly at the threshold level. The optimal profit is increasing and piecewise concave in the inventory of old product.

Observation 2. The optimal threshold level increases in demand standard deviation. The decrease in optimal order quantity at the threshold level increases in demand standard deviation. The optimal profit decreases in demand standard deviation.

Remark 1. We also considered a model of partial discounting where the retailer decides how many of the old units to discount at δ. This contrasts with Model A which considers a binary decision whether to discount all the old units or none at all. Interestingly, we find that the partial discounting policy is never optimal. That is, it is optimal for the retailer to discount all or nothing.

4.2. Model B

The difference in Model B from Model A is that instead of an *all-or-nothing* discount in which either all or none of the customers prefer old units to new ones, we now consider discounts x so that a fraction $\alpha(x)$ prefers old units. For simplicity, we define $\alpha(x) \equiv ax$ in which a measures the *discount sensitivity* of customers. Next, we describe the results from our analysis of Model B.

4.2.1. Structure of Optimal Policy

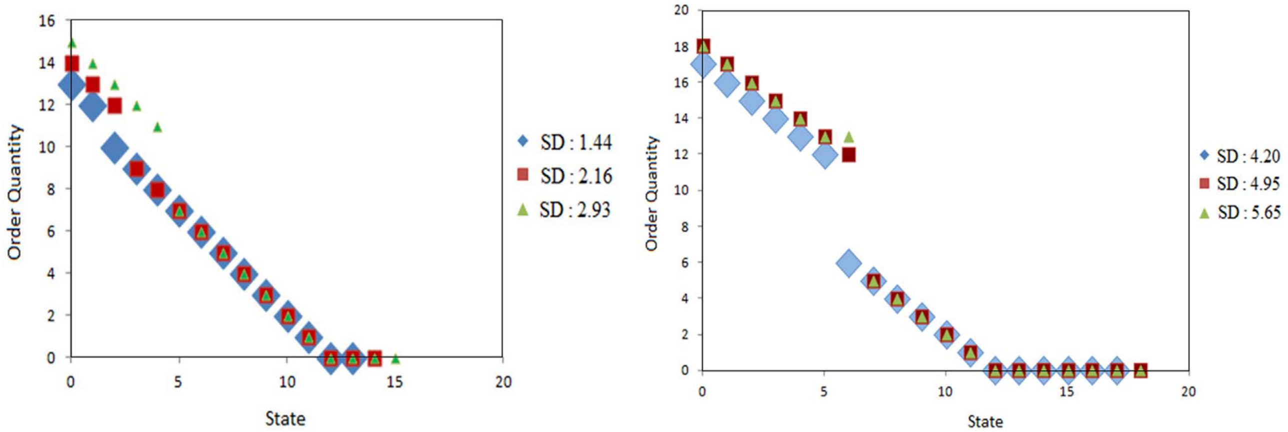
As in Section 4.1, if there is some discounting, we find that the optimal profit is an increasing piecewise concave function (with two pieces) in the inventory of the old product for different types of demand distributions and parametric values. Further, the order quantity also decreases with more old units. However, Fig. 8a shows that the optimal discount increases first and then decreases (to zero). This non-monotonicity of discount in the inventory of old product s_t is explained as follows. There are two effects of discounting in period t: (i) loss in profit in that period from discounting old units and (ii) gain in profit in the next period, i.e., period $t-1$, from having a higher inventory (because π_{t-1} is increasing in s_{t-1}). The loss in profit is high when s_t is high because more units are discounted on average, while the gain in profit is low when s_t is low because s_{t-1} does not increase much from discounting more; these factors explain why discount is low (or zero) when s_t takes low or high values. On the other hand when s_t is intermediate, a high discount is offered.

In Figure 8a, when the discount decreases, it reduces immediately to zero. However, that is not always true. Fig. 8b shows a scenario in which discount reduces from state 1 to state 2 but does not become zero. For this reason, we refer to the *threshold* in Model B as the state at which the discounts stops increasing and starts reducing (e.g., the threshold in Fig. 8b is state 1). Next, we examine how the optimal discounting policy changes with a key parameter of Model B: the discount sensitivity a.[8]

[6] When there are too many old units, the retailer does not discount because she incurs too much loss from doing that. She would rather reduce the order quantity significantly.

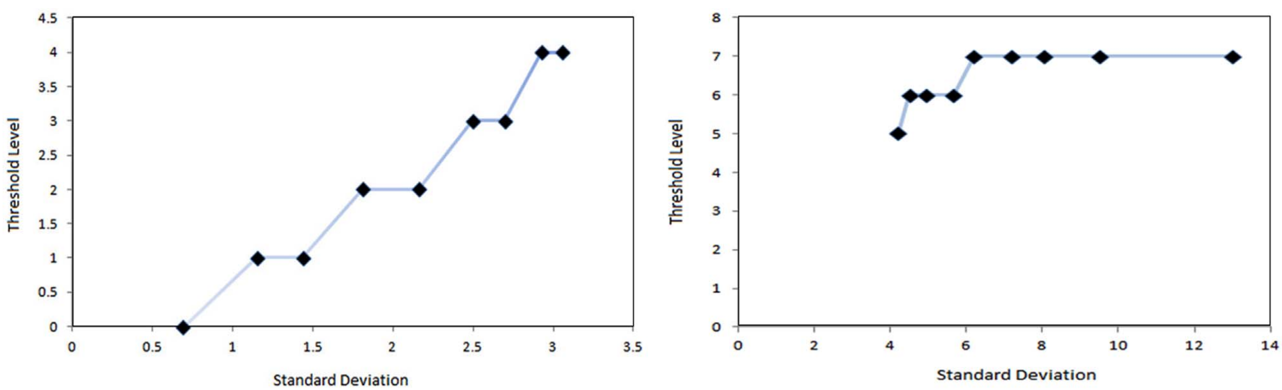
[7] Note that the cost for this inventory is already *sunk* and does not affect the profit.

[8] For conciseness, we do not discuss about sensitivity analysis with respect to cost c because we already consider it with Model A in Section 4.1 and we find that the results here are similar based on our numerical examples.



(a) Binomial demand distributions (b) Negative-binomial demand distributions

Fig. 4. Optimal order quantity for various demand distributions with fixed mean 12.5, (a) Binomial demand distributions, (b) Negative-binomial demand distributions.



(a) Binomial demand distributions (b) Negative-binomial demand distributions

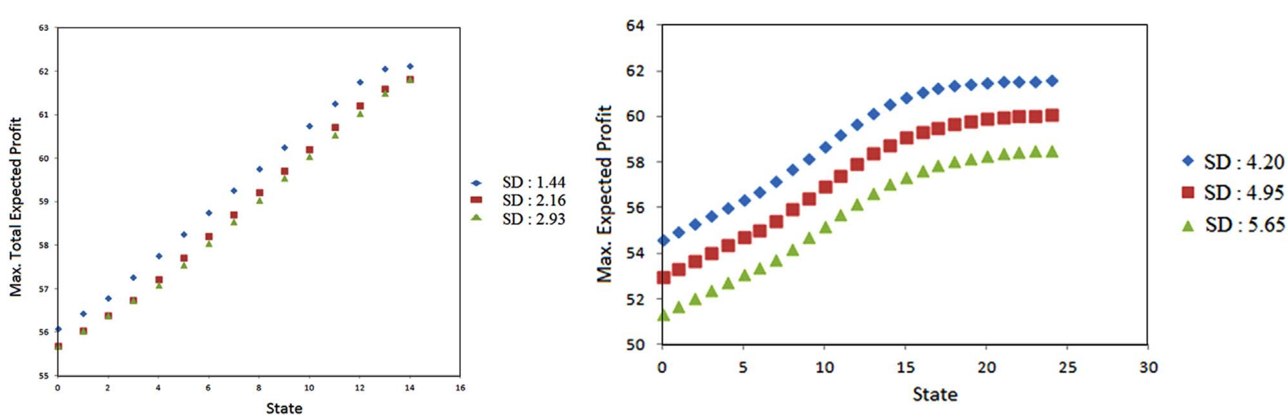
Fig. 5. Optimal threshold level for various demand distributions with fixed mean 12.5, (a) Binomial demand distributions, (b) Negative-binomial demand distributions.

4.2.2. Sensitivity analysis

Fig. 9 shows the optimal discounting policies under different values of a. We find that as a increases, the threshold increases. As customers become more sensitive to discounts, the retailer offers discounts under more states. However, we note that for a given state, *the discount itself may increase or decrease* in a. For instance, when a increases from 1 to 3, the discount decreases from 14% to 6% in state 1 while it increases from 0% to 10% in state 2. Next, we compare Model B with Model A.

4.2.3. Comparison of model B with model A

In order to make a meaningful comparison between the two models, we consider examples with $\delta = 1/a$ (so that when a discount of δ is offered, all the customers prefer old units) and all other parameters being identical for the two models. Fig. 10 shows the optimal profits in Models A and B for one such example. We find that the optimal profit in Model B exceeds that of Model A because the discounting decisions in Model A (discounts of zero and δ) can be easily



(a) Binomial demand distributions (b) Negative-binomial demand distributions

Fig. 6. Optimal profit for various demand distributions with fixed mean 12.5, (a) Binomial demand distributions, (b) Negative-binomial demand distributions.

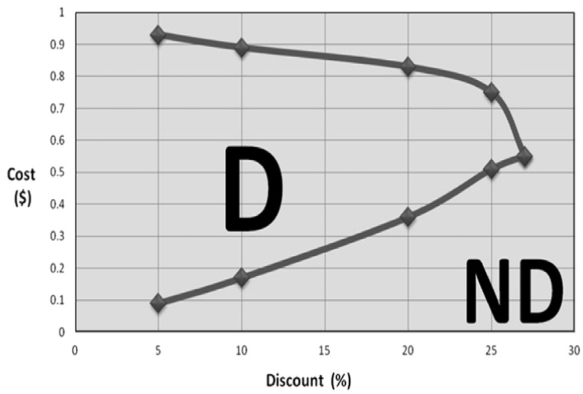


Fig. 7. Impact of procurement cost and discount on optimal discounting policy with $D_t \sim U[0, 9]$.

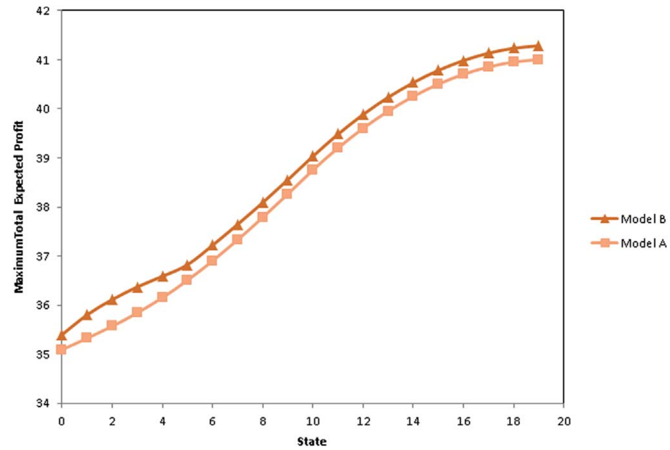


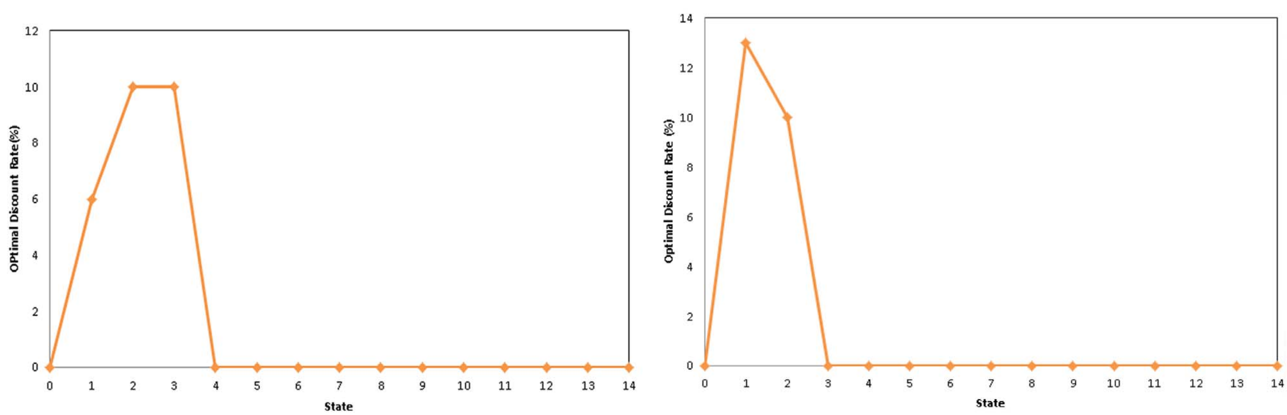
Fig. 10. Optimal profit in Model A and Model B; δ = 33%, a=3, c=0.5, and $D_t \sim U[0, 19]$.

reciprocated in Model B (with discounts of zero and $1/a$ respectively). However, we find that the difference in optimal profits between the two models is not significant (less than 2%). We find that this difference is also insignificant for examples with other parametric values and demand distributions, the details of which we omit for conciseness.

Fig. 11 shows the corresponding optimal discounting policies in the two models. We find that discounts are offered in more states under Model B than in Model A. This result is explained as follows: the flexibility to offer lower discounts (than δ) enables the retailer to give a discount in Model B even when the inventory of old product is higher than the threshold for Model A. Based on our numerical results, we find that the threshold value for Model B is always greater than or equal to the threshold for Model A.

Based on the results presented above and those from other numerical examples, we summarize the following observation about Model B:

Observation 3. The optimal discount first increases and then decreases (eventually to zero) in the inventory of old product. As the discount sensitivity a increases, (i) the threshold increases but (ii) the discount for a given state may increase or decrease. Finally, we find by comparing Model B with Model A that (i) the increase in profit in Model B is generally not significant and (ii) the threshold in Model B is higher.



(a) $a = 3,\ c = 0.5$ (b) $a = 5,\ c = 0.45$

Fig. 8. Optimal discounting policy with $D_t \sim U[0, 14]$, (a) a=3, c=0.5, (b) a=5, c=0.45.

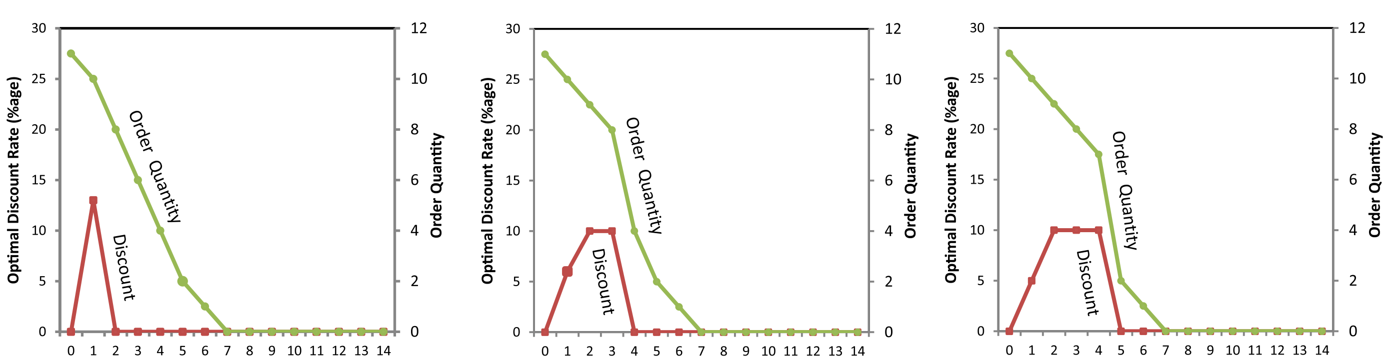


Fig. 9. Variation of optimal order quantity and discount with inventory of old product for a=1, 3, and 5 (from left to right).

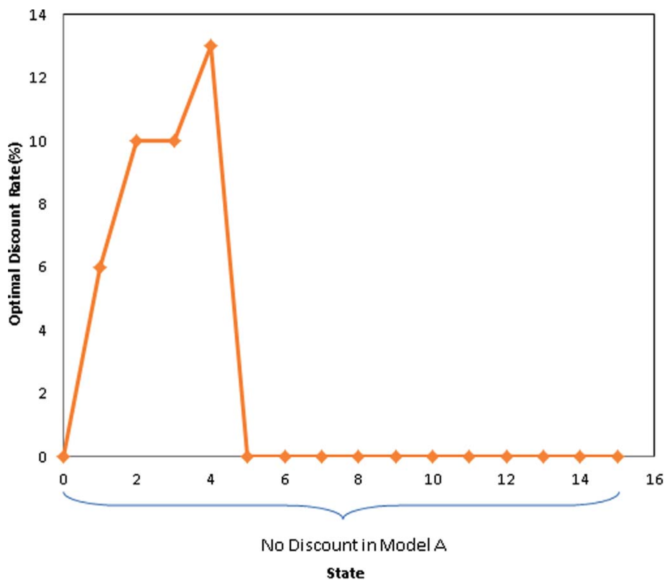


Fig. 11. Optimal discounting policies in Model A and Model B; $\delta = 33\%$, a=3, c=0.5, and $D_t \sim U[0, 15]$.

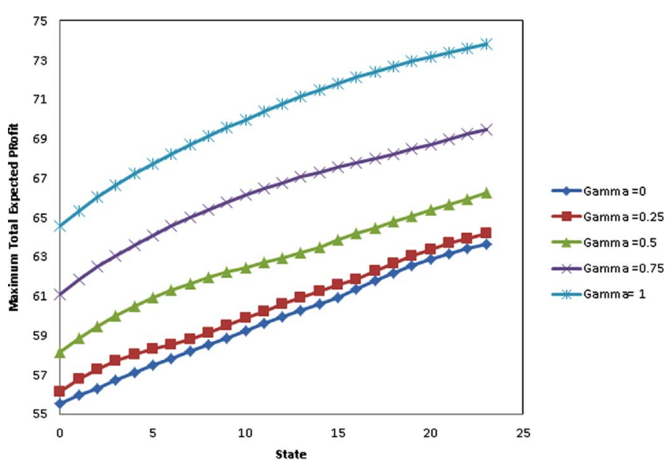


Fig. 12. Variation of optimal profit with inventory of old product, for different γ's.

4.3. Model C

The difference in Model C from Model B is that there is now a new pool of customers, whose size is given by $bx\alpha(x)D$ in which $b > 0$ is a measure of *advertising effectiveness* (about the discounted sale of the old product), x is the discount, and $\alpha(x)D$ is the number of customers from the original demand who prefer the old product to new product. We parameterize the fraction of this new pool of customers, the *early-bird bargain hunters*, who get the old product before those from the original demand by γ. Next, we discuss about the results from our analysis of Model C.

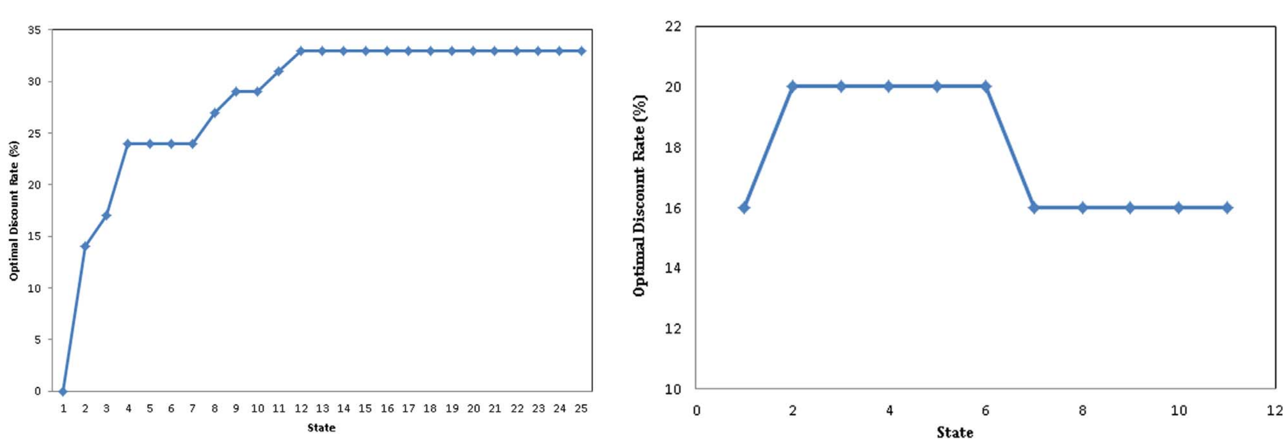
4.3.1. Structure of optimal policy

Fig. 12 shows an example of how the optimal profit changes in Model C. As in Models A and B, it is increasing in the inventory of old product; however, unlike those models, it is now concave and smooth instead of being piecewise concave with two pieces when γ is too low or high (e.g., $\gamma = 0, 1$ in the figure). Fig. 13a shows how the optimal discount changes with the inventory of old product. Again, we find that, unlike in Model B in which it first increases and then reduces to zero (see Figs. 8a and 8b), it is always increasing. This result can be explained as follows: with $\gamma = 1$ and b=3, the significant amount of new pool of customers makes it profitable for the retailer to offer discount even when the inventory of old product is high. While this result is generally true, it is not always the case. Even with such high γ and b values, the demand distribution is also important. Fig. 13b shows a scenario in which the optimal discount rate increases and then decreases for a *general demand distribution*. Next, we analyze how the optimal policy and profits change with two key parameters: advertising effectiveness b and the fraction of early-bird bargain hunters γ.

4.3.2. Sensitivity analysis

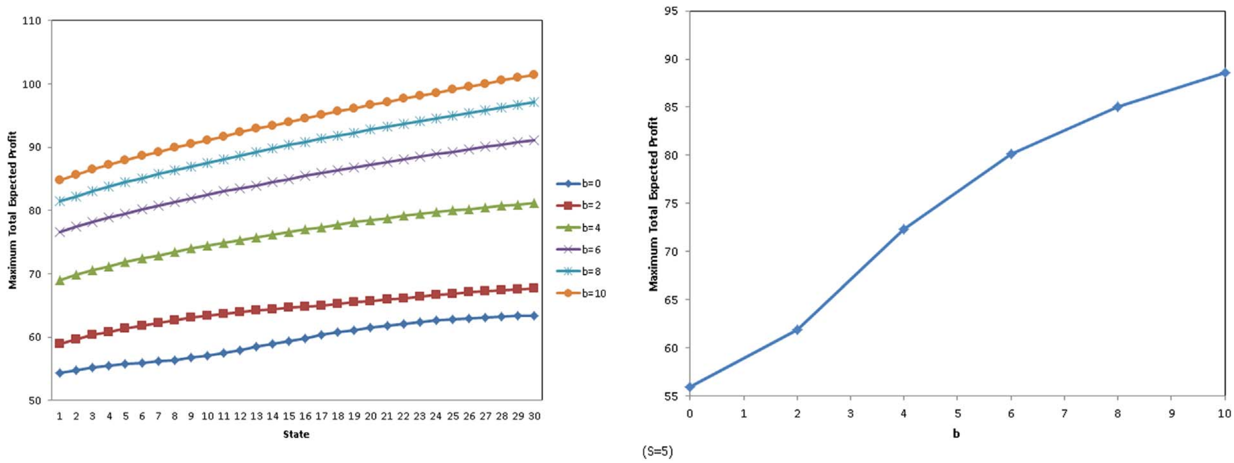
First, we first consider the variation of b. Fig. 14a shows how the optimal profit changes with the inventory of old product under different values of b. We find that it is increasing, and it becomes a piecewise concave function (with two pieces) for low values of b while it is a smooth concave function for high values of b. That is because as $b \to 0$, Model C becomes equivalent to Model B. Fig. 14b illustrates how the optimal profit for a given state (the inventory of old product is five units) changes with b. We find that it is convex (concave) in b when b is low (high), which indicates *increasing returns* (*diminishing returns*). Basically, increasing the advertising effectiveness is much more beneficial when it is low than when it is high.

Fig. 15a demonstrates how the optimal discount changes with the inventory of old product under different values of b. We find that the variation of optimal discounting policy has a complicated pattern. For low values of b, Model B becomes similar to Model C, and as the inventory of old product increases, the optimal discount first increases



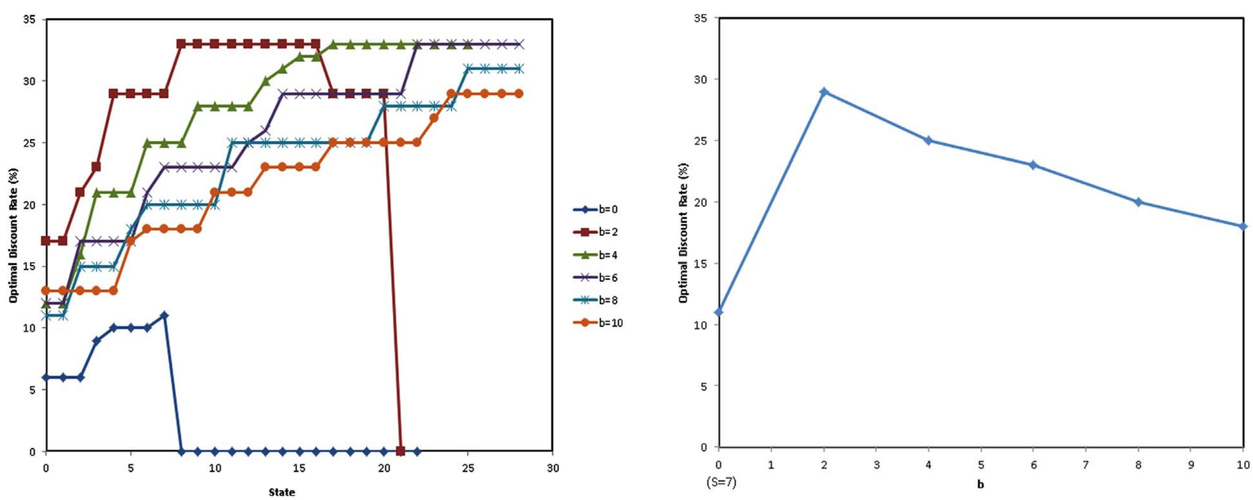
(a) Discount monotonically increases (b) Discount first increases and then decreases

Fig. 13. Optimal discounting policy as inventory of old product increases, (a) Discount monotonically increases, (b) Discount first increases and then decreases.



(a) Variation of optimal profits with inventory of old product (b) Optimal profit when inventory of old product is five units

Fig. 14. Optimal profits for different values of *b*, (a) Variation of optimal profits with inventory of old product, (b) Optimal profit when inventory of old product is five units.



(a) Variation of optimal discount with inventory of old product (b) Optimal discount when inventory of old product is seven units

Fig. 15. Optimal discount for different values of *b*, (a) Variation of optimal discount with inventory of old product, (b) Optimal discount when inventory of old product is seven units.

and then decreases to zero. For high values of b, the optimal discount is always increasing. In our numerical examples, we find that this pattern is robust and holds across different values of γ, c, and different distributions. Fig. 15b shows that the optimal discount for a given state (the inventory of old product is seven) is non-monotone in b: it first increases and then decreases. As the advertising effectiveness increases, initially, more discount is offered to attract even more customers from the new pool to buy the old product. However, when it increases even further, less discount is sufficient to obtain the required amount of new pool of customers. Finally, Fig. 16 shows that the order quantity is increasing in b for a given state (the inventory of old product is five) because as b increases, selling the old product becomes more likely and that enables the retailer to order more quantity.

Second, we consider how the fraction of early-bird bargain hunters, γ, affects the optimal profit and discounting policy. Fig. 17a describes how the optimal profit changes with the inventory of old product under different values of γ. We find that when γ is very low ($\gamma \approx 0$) or very high ($\gamma \approx 1$), the optimal profit is smooth and concave in the inventory of old product; however, it is piecewise concave for intermediate values of γ. Fig. 17b shows that the optimal profit for a given state (the inventory of old product is zero) is increasing in γ which indicates that

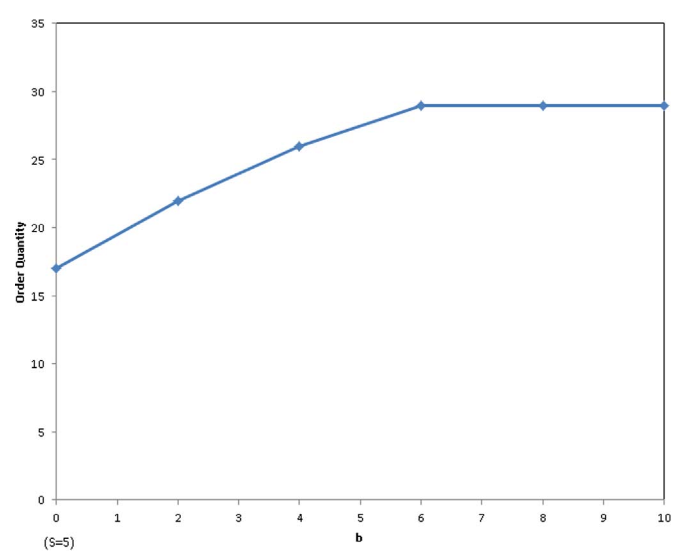
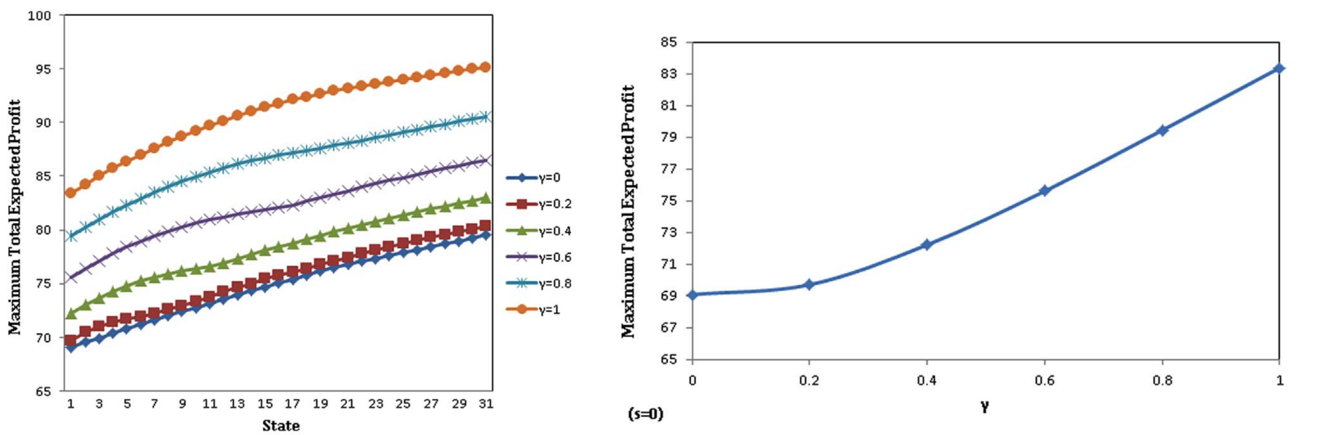


Fig. 16. Optimal order quantity, when inventory of old product is five units, for different values of *b*.



(a) Variation of optimal profits with inventory of old product

(b) Optimal profit when inventory of old product is zero

Fig. 17. Optimal profits for different values of γ, (a) Variation of optimal profits with inventory of old product, (b) Optimal profit when inventory of old product is zero.

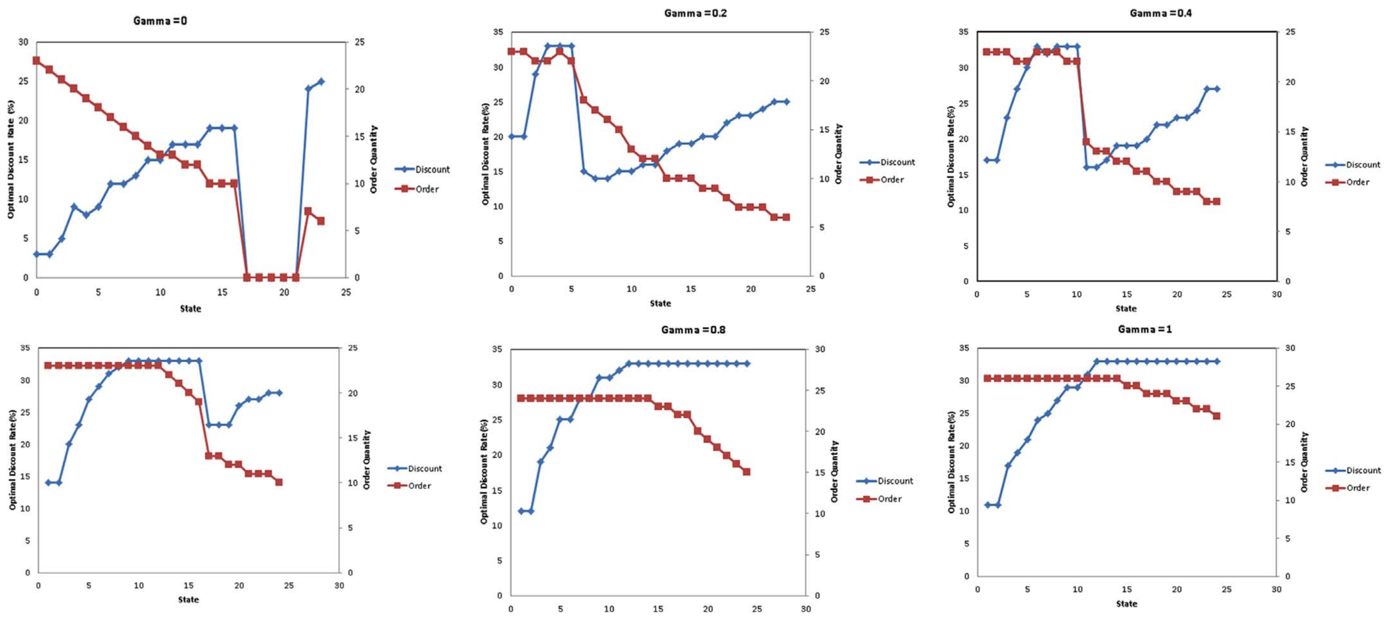


Fig. 18. Variation of optimal discount and order quantity with the inventory of old product, under different values of γ.

more early-bird bargain hunters in the new pool benefit the retailer. Further, the increase in profit is higher for higher values of γ which shows *increasing returns* in the optimal profit vis-a-vis γ. This result is opposite to the diminishing returns we observe earlier for the advertising effectiveness b, and it indicates that increasing the proportion of "early-bird" customers is much more important.

Fig. 18 shows how the optimal discount and order quantities change with the inventory of old product, under different values of γ. We find two results to be interesting. First, we find that order quantity can *increase* with the inventory of old product (when $\gamma = 0$). This result is fundamentally different from what we observe in Models A and B in which it always decreases with the inventory of old product. The retailer *discounts more and also increases the order quantity even with a higher inventory due to the presence of a significant new pool of customers who only want the old product*. Second, we find that the variation of optimal discount with inventory exhibits two types of patterns. It is increasing when γ is high, which is explained by the significant benefit from the new pool of customers, which makes the retailer to keep offering more discounts with higher inventory. It first increases, then decreases, and again increases in the inventory of old

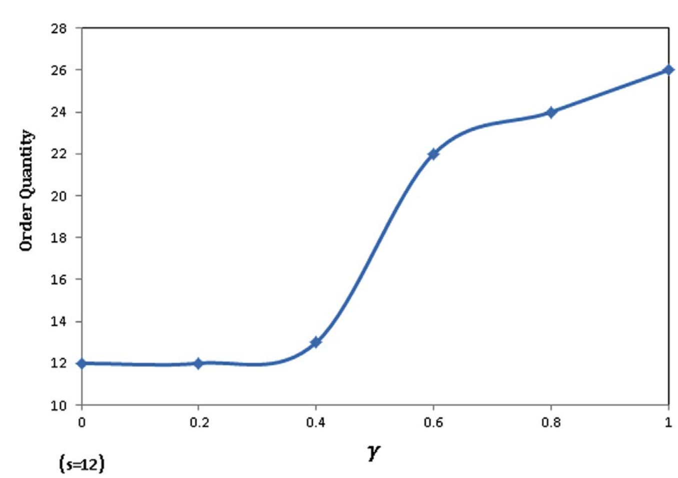
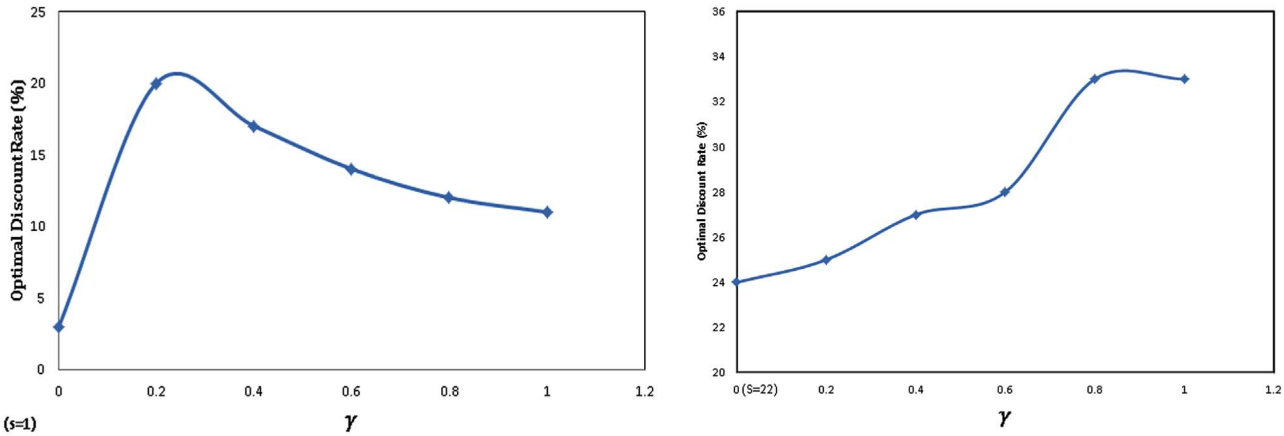


Fig. 19. Optimal order quantity, when inventory of old product is 12, for different values of γ.



(a) Inventory of old product $s = 1$ (b) Inventory of old product $s = 22$

Fig. 20. Variation of optimal discount with γ, (a) Inventory of old product s=1, (b) Inventory of old product s=22.

product. The initial increase and the decrease later are akin to what happens in Model B. However, when the inventory increases further, the optimal discount actually *increases*. That is because the new pool of customers makes more discounting profitable by driving more customers from the base demand (who prefer old product to new product but find it out of stock) toward the new product.

Fig. 19 shows how the optimal order quantity, when the inventory of old product is 12, changes with γ. It is always increasing in γ because the retailer orders more with higher fraction of early-bird customers from the new pool. Further, it is S-shaped and increases much more for intermediate values of γ than when γ is low or high. Figures 20a and 20b show how the optimal discount changes for a given state: inventory of old product are 1 unit and 21 units respectively. In Fig. 20a, in which the inventory is low, the optimal discount first increases and then decreases in γ. This trend is explained as follows: when γ is low, the retailer takes advantage of an increase in γ by offering more discount and increasing the new pool of customers as well; however, when γ is high, because the inventory is low, the retailer reduces the discount as γ increases. In other words, when the inventory of old product is low, γ and the discount are *compliments* when γ is low but they become substitutes under high values of γ. In Fig. 20b, because the inventory is high, discount and γ always act as *compliments* in helping the retailer sell the old product, and so the optimal discount is increasing in γ.

Based on the analysis in Section 4.3, we make the following key observations.

Observation 4. The optimal order quantity can increase in the inventory of old product due to the presence of a new pool of customers. The optimal discount follows three kinds of patterns: (i) it increases in the inventory, (ii) it first increases and then decreases, or (iii) it initially increases, then decreases, and and again increases in the inventory. The last one is different from what is observed in Model B and is explained by the presence of new pool of customers.

Observation 5. As the advertising effect b increases, the optimal values (for a given inventory) show the following trends. The optimal order quantity and profit are both increasing with the latter forming a S-curve. However, the optimal discount can be non-monotone; it first increases and then decreases (e.g., see Fig. 15b).

1 Period Old (S1) \ 2 Period Old (S2)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	* ND 13 D2 13 [	* D2 12 (	* D2 11 (	* D2 10	* D2 9 D:	* D2 8 D	* D2 7 D	* ND 3	* ND 2 D	* ND 0_1	* ND 0 D	* ND 0 C	* ND 0 [	* ND 0 D	* ND 0 D	* ND 0 D:	* ND 0 D	* ND 0 D	* ND 0 D1	* ND 0 D1
1	* ND 12 D2 12	D2 11	D2 10	D2 9	D2 8	D2 7	ND 3	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
2	* ND 10 D2 10	D2 9	D2 8	D2 7	D2 6	D2 5	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
3	* ND 9 D2 9	D2 8	D2 7	D2 6	D2 5	D2 4	ND 1	ND 0	ND 0	ND 0	ND 0	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
4	* ND 8 D2 8	D2 7	D2 6	D2 5	D2 4	ND 1	ND 0	ND 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
5	* ND 7 D2 7	D2 6	D2 5	D2 4	D2 3	ND 0	D2 1	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
6	* ND 6 D2 6	D2 5	D2 4	D2 3	D2 2	D2 1	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
7	* ND 5 D2 5	D2 4	D2 3	D2 2	D2 1	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
8	* ND 4 D2 4	D2 3	D2 2	D2 1	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0_2	* D2 0_2	* D2 0_2	* D2 0 D
9	* ND 3 D2 3	D2 2	D2 1	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0_1	* D2 0_1	* D2 0_1	* D2 0_1	* D2 0_1
10	* ND 2 D2 2	D2 1	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
11	* ND 1 D2 1	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
12	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
13	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
14	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
15	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
16	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
17	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
18	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	* D2 0 [	* D2 0 D	* D2 0 C	* D2 0 C	* D2 0 D:	* D2 0 D
19	* ND 0 D2 0 D:	* D2 0 C	* D2 0 C	* D2 0 C	* D2 0 D	* D2 0 C	* D2 0 [	* D2 0	* D2 0 C	* D2 0 (	* D2 0 C	* D2 0 [	* D2 0 (	* D2 0 [	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0

Fig. 21. Optimal policy for 3-period shelf life with $D_t \sim Bin(20, 0.5)$.

2 Period Old (S2) / 1 Period Old (S1)	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
0	* ND 2	* D2	* D2 1:	* D2 :	* D2	* D2	* ND	* ND	* ND	* ND	* ND 0	* ND (	* ND 0	* ND 0	* ND 0	* ND 0 D	* ND 0 (	* ND 0 D	* ND 0 D:	* ND 0 D
1	* ND 1	D2 18	D2 17	D2 16	ND 6	ND 4	ND 3	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
2	* ND 1	D2 15	D2 14	D2 13	ND 5	ND 3	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
3	* ND 1	D2 12	D2 11	ND 5	ND 4	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
4	* ND 1	D2 10	ND 6	ND 4	ND 3	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
5	* ND 1	D2 9	ND 5	ND 3	ND 2	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
6	* ND 9	D2 8	ND 4	ND 2	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
7	* ND 8	D2 7	ND 3	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
8	* ND 7	D2 6	ND 2	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0
9	* ND 5	D2 4	ND 1	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
10	* ND 4	D2 3	ND 0	ND 0	ND 0	ND 0	ND 0	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
11	* ND 3	D2 2	ND 0	ND 0	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
12	* ND 2	D2 1	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
13	* ND 1	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
14	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
15	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
16	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
17	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
18	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
19	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	* D2 0	* D2 (	* D2 0	* D2 0	* D2 0	* D2 0 (	* D2 0	* D2 0 |	* D2 0 D	* D2 0 D
	* ND 0	* D2	* D2 0	* D2 (	* D2	* D2	* D2	* D2	* D2	* D2	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0	ND 0

Fig. 22. Optimal policy for 3-period shelf life with $D_t \sim NB(7, 0.8)$.

Observation 6. As the fraction of early-bird (bargain hunters) customers γ increases, the optimal values (for a given inventory) show the following trends. The optimal order quantity is increasing and convex while the the optimal profit is increasing and forms a S-curve. The optimal discount follows a complex pattern: if the inventory is low it first increases and then decreases in γ, while if the inventory is high it is monotonically increasing in γ.

4.4. Model D

The difference in Model D from Model A is that shelf life need not be 2 periods, but n periods in general. To examine how longer shelf life affects our earlier results, we consider n=3 and report our findings. We also illustrate how Model A can be used to approximate Model D when n=4.

4.4.1. Structure of optimal policy

Unlike Model A, an n-period shelf life requires the retailer to keep track of more than one state variables in each period, which complicates the analysis of the dynamic program. For illustration, we consider n=3 which results in two state variables; namely, the inventory of one-period old products and the inventory of two-period old products. For the various demand distributions we considered (uniform, binomial, negative binomial), we find that the optimal order quantity, discounting policy and profit do not behave as in Model A with respect to the total inventory of old products. This is because the age mix of the old products also affects the optimal decisions. Moreover, for a fixed value of either state variable, the optimal discounting policy is also not a threshold policy in the other state variable. Figs. 21 and 22 provide examples for binomial and negative binomial demand distributions, respectively. We observe similar patterns for $n > 3$.

4.4.2. Model A as approximation

In this subsection, we consider a product with a 4-period shelf life. Here, the optimal solution can be obtained by solving Model D. However, suppose the retailer restricts herself (or is restricted) to make decisions only in alternating periods, say periods $T, T-2, T-4, \ldots$. During the other periods, she does not place any order, i.e. q_t=0, and she does not change her discounting decision, i.e. $\mathbf{y_t} = \mathbf{y_{t+1}}$. The problem can still be solved using (5) in Model D with some modification. Specifically, it can be modeled as follows.

$$\pi_t(\mathbf{s_t}) = \max_{q_t \geq 0, \mathbf{y_t} \in \{0,1\}^3, (q_t, \mathbf{y_t}) \in \Lambda(\mathbf{y_{t+1}})} \{pE[\min(q_t + \sum_{i=1}^{3} s_{t,i}, D_t)]$$
$$- \delta E[\min(\sum_{i=1}^{3} s_{t,i} y_{t,i}, D_t)]$$
$$-cq_t + E[\pi_{t-1}(([q_t, s_{t,1}, s_{t,2}]' - \mathbf{M_t})^+)]\}, \forall\, t > 0;$$
$$\pi_0(.) = 0. \qquad (6)$$

where $\Lambda(\mathbf{y_{t+1}}) = \{(q_t, \mathbf{y_t}) | q_t = 0, \mathbf{y_t} = \mathbf{y_{t+1}} \text{ if } t \in \{T-1, T-3, T-5, \ldots\}\}$.

While Model (6) maintains a 3-dimensional state space, we propose an equivalent formulation using only a one-dimensional state space. For ease of exposition, we assume T is even.

Theorem 1. *Model (6) is equivalent to (1) in Model A with T replaced by $T' = T/2$ and D_t replaced by $D'_t = D_{2t} + D_{2t-1}$ with cdf Φ', the* 2*-fold convolution of Φ.*

Proof. Observe that $\mathbf{s_t} = [s_{t,1}, s_{t,2}, s_{t,3}]' = ([q_{t+1}, s_{t+1,1}, s_{t+1,2}]' - \mathbf{M_{t+1}})^+$. At each decision epoch $t \in \{T-1, T-3, T-5, \ldots\}$, we know that $s_{t,1} = 0$ because $q_{t+1} = 0$. Because $t + 2 \in \{T-1, T-3, T-5, \ldots\}$, we also have $s_{t+2,1} = 0$. This implies that $s_{t+1,2} = 0$. Hence, for any period

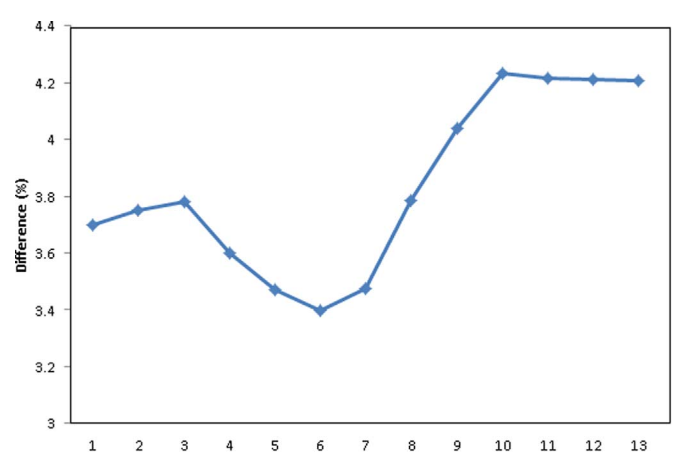


Fig. 23. Optimality loss for Model A approximation for n=4, c=0.5, $\delta = 0.15$ and $D_t \sim U[0, 7]$.

$t \in \{T-1, T-3, T-5, \ldots\}$, only $s_{t,2}$ is possibly nonzero. This means that at each decision epoch, there is only one relevant state variable. Then it is easy to see that this problem is equivalent to Model A with number of decision epochs half the number of periods, the duration of each epoch is two periods, and the demand between consecutive epochs is the sum of demands from two consecutive periods. □

This result is meaningful because it allows us to solve a 4-period shelf life problem using a 2-period shelf life approximation. Naturally, we want to know how much is the optimality loss due to such approximation. In Fig. 23, we find that the reduction in profit is not substantial (e.g. it does not exceed 4.3%). This finding is interesting because it implies that Model A is a sufficient approximation for Model D. Furthermore, it provides justification for assuming n=2 in Models B and C as the optimality loss is sufficiently low to justify the insights one can obtain from these 2-period models.

Based on the results presented above, we summarize the following observation about Model D:

Observation 7. The structure of the optimal policy observed in Model A no longer holds for Model D. Nonetheless, Model A can be a good approximation for Model D with minimal optimality loss.

5. Conclusion

In this paper, we study a periodic-review inventory problem for a perishable product with limited shelf life. In addition to replenishment decisions, the retailer can offer a discount to attract customers to older units. We consider four models with different customer characteristics. In the base model with shelf life of two periods and an exogenous discount, we find that the optimal discounting policy is a threshold policy while the optimal order quantity decreases in the inventory of old units with a significant decrease at the threshold. We also investigate the effect of the degree of demand uncertainty on the optimal decisions. Moreover, we find that partial discounting where the retailer decides on the number of old units to discount is never optimal. Extending this model to allow the retailer to decide on the discount subsequently affecting the fraction of customers preferring old units, we find that the optimal discount first increases and then decreases (eventually to zero) in the inventory of old units, while the threshold increases in discount sensitivity. Compared to exogenous discount, the increase in profits in generally not significant and the threshold is higher. Extending the model further to allow a new pool of customers to be attracted solely to the old units, we also consider the effect of two new parameters; namely, advertising effectiveness and the fraction of bargain hunters (new customers who beat the original customers to the old products). In this setting, the optimal order quantity increases in the inventory of old units due to the presence of new customers. As advertising effectiveness increases, optimal order quantity and profit increases while the optimal discount can be non-monotone. As fraction of bargain hunters increases, optimal order quantity increases in a convex fashion, the optimal profit also increases while the optimal discount behaves in certain patterns according to whether the inventory of old units is low or high. Finally, extending the base model to shelf life longer than two periods, we observe that the structure of the optimal policy no longer holds. However, we find that the base model can be used to approximate say a four-period problem with minimal optimality loss. This suggests that one can learn insights from two-period models without too much reduction in profits. Nonetheless, the option to set the discount and the presence of bargain hunters for products with shelf life longer than two periods are also interesting to study and analytically challenging to characterize optimally. We leave this issue to future research.

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